A CLASSICAL BOUND ON QUANTUM ENTROPY

$$\boxed{0 \leq S_q \leq \ln\left(\frac{e\sigma^2}{2\hbar}\right)}$$

involving the variance σ^2 in phase space of the **classical** limit distribution of a given arbitrary **quantum** system. No Hamiltonian required.

An **upper bound on the lack of information**. Black Hole entropic behavior: collective flow of information in need of robust estimates. $\rightsquigarrow$ Through gross geometrical and semiclassical features of the system—instead of toilsome, subtler, detailed accounts of quantum states.

$\rightsquigarrow$ **Combines** upper bound for the entropy of classical continuous distributions (Shannon, 1949) with classical limit of intricate quantum systems in phase space (Braunss 1994). Tracks the information loss involved in smearing away quantum effects. $\rightsquigarrow$ The quantum entropy of a system is majorized by that of its 'ignorant' classical limit.

The sum over states gives the von Neumann entropy for a density matrix,

$$0 \leq S_q = -\mathrm{Tr}\ \rho\ \ln \rho = -\langle \ln \rho \rangle\ .$$

$\Rrightarrow$ Transcribes in phase space through the Wigner transition map to

$$\boxed{0 \leq S_q = -\int dxdp\ f\ \ln_\star(hf)\ ,}$$

where Groenewold's (1946) $\star$-product, $\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}$, serves to **define** $\star$-functions.

Braunss has argued that, for $S_q + \ln h \to S_{cl}$, and as Planck's $\hbar \to 0$,

$$\boxed{0 \leq S_q \leq S_{cl} - \ln h\ .}$$

$\pitchfork$ This upper bound reflects the **loss of quantum information** involved in the smearing implicit in the classical limit.

$\leadsto$ Combined with Shannon's bound, this now yields the inequality proposed: the entropy is bounded above by an expression involving the variance of the corresponding **classical limit distribution function**.

⊛ The quantum entropy is recognized as an expansion

$$S_q = \sum_{n=1}^{\infty} \frac{\langle (1-\rho)^n \rangle}{n} = \sum_{n=1}^{\infty} \frac{\langle (1-hf)_\star^n \rangle}{n}.$$

The leading term, $n = 1$, $1 - \mathrm{Tr}\rho^2 = \langle 1 - hf \rangle$, is the **impurity**. Like the entropy itself, it **vanishes for a pure state**, for which $\rho^2 = \rho$, or, equivalently, $f \star f = f/h$. Each term in the expansion projects out ρ, or $\star hf$, respectively, so **pure states saturate the lower bound on S_q**.

• **Illustration** by the elementary physics paradigm of a Maxwell-Boltzmann thermal bath of oscillator excitations of one degree of freedom: its phase-space representation happens to be a (maximal entropy $\sim$ chaos $\sim$ disorder) Gaussian. The Gaussian Wigner Function of **arbitrary** half-variance E,

$$f(x,p,E) = \frac{e^{-\frac{x^2+p^2}{2E}}}{2\pi E} = e^{-\frac{x^2+p^2}{2E} - \ln(2\pi E)}.$$

(Oscillator with mean energy $E = \langle \frac{x^2+p^2}{2} \rangle$.)

For $E = \hbar/2$, the distribution reduces to just f_0, the Wigner Function for a **pure state** (the ground state of the harmonic oscillator). ↬

$f_0 \star f_0 = \frac{f_0}{h}$. Since f_0 is $\star$-orthogonal to each of the terms in the sum, $S_q = 0$: saturation of the maximum possible information content.

For generic width E, the Wigner Function f is not that of a pure state, but **it still happens to always amount to** a $\star$-exponential ($e_\star^a \equiv 1 + a + a \star a/2! + a \star a \star a/3! + \ldots$) as well,

$$hf = e^{-\frac{x^2+p^2}{2E} + \ln(\hbar/E)} = e_\star^{-\frac{\beta}{2\hbar}(x^2+p^2) + \ln(\frac{\hbar}{E}\cosh(\beta/\hbar))},$$

where an "inverse temperature" variable $\beta(E,\hbar)$ is useful to define,

$$\tanh(\beta/2) \equiv \frac{\hbar}{2E} \leq 1 \qquad \Longrightarrow \qquad \beta = \ln \frac{E+\hbar/2}{E-\hbar/2} \ .$$

$$\boxed{S_q(E,\hbar) = \frac{E}{\hbar} \ln\left(\frac{2E+\hbar}{2E-\hbar}\right) + \frac{1}{2}\ln\left((\frac{E}{\hbar})^2 - \frac{1}{4}\right) = \frac{\beta}{2}\coth(\beta/2) - \ln(2\sinh(\beta/2)).}$$

$\looparrowright$ **monotonically nondecreasing** function of E, attaining the lower bound 0 for the pure state $E \to \hbar/2$ ($\beta \to \infty$, zero temperature).

The classical limit, $\hbar \to 0$ ($\beta \to 0$, infinite temperature) follows,

$$S_q \to 1 + \ln(E/\hbar) = \ln(\pi e 2E) - \ln h = S_{cl}(E) - \ln h \ .$$

Explicitly seen to bound the above expression for all E; saturates it for large $E >> \hbar$, in accordance with Braunss' bound. $\looparrowright$ the upper bound is **saturated** for Gaussian quantum Wigner functions with $\sigma^2 >> \hbar$.

$\divideontimes$ The region $E < \hbar/2$, corresponding to ultralocalized spikes excluded by the uncertainty principle, was **not allowed** by the above derivation method, since, in this region, no $\star$-**Gaussian can be found** to represent the Gaussian. (It would amount to complex β and S_q.)

$\pitchfork$ Applications in holographic BH physics and quantum computing; LHC contact with gravitational physics confronting quantum randomness. Compton wavelength invisible inside its own Schwarzschild horizon.