Fermion Localization in Compact Extra-Dimensions *

by

Manuel Toharia

*ANL, May 2004

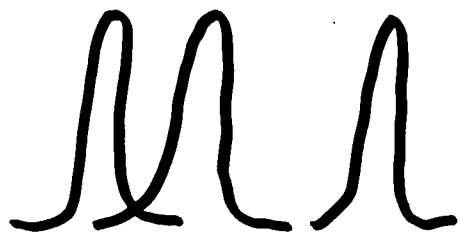
OUTLINE

- Introduction
- ϕ^4 theory in 5D
- Fermion Localization
- Conclusion

INTRODUCTION

ARKANI-HAMED
SCHMALZ

IDEA:

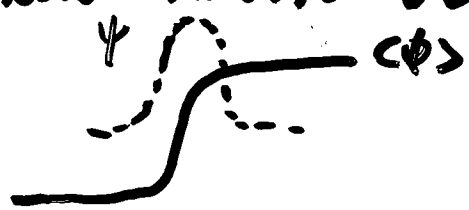


- FERMION ZERO MODES 'LOCALIZED' AROUND DIFFERENT POINTS IN THE X-TRA DIMENSION

  ⇒ SMALL OVERLAP WILL SUPPRESS FERMION INTERACTION TERMS IN 4D EFF. LAGRANGIAN

  - YUKAWA COUPLINGS EXPLAINED BY GEOGRAPHY
  - SUPPRESS PROTON DECAY OPERATORS

- NON-TRIVIAL VEV OF BULK SCALAR FIELD USED:



$\langle\phi\rangle \mp 4$

KAPLAN - TAIT INTRODUCE Z_2 SYMMETRY TO AVOID STABILITY PROBLEM

POTENTIAL OF ϕ

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2$$

GEORGI, GRANT, HAILU SOLVE THE PROBLEM 'EXACTLY' FOR $\langle\phi\rangle \sim \tanh y \Longleftarrow$ ∞ X-TRA DIM. THEN GET APPROXIMATE SOLUTION IN THE LIMIT $L \gg \frac{1}{m}$

WE INTEND TO SOLVE THIS PROBLEM WITH NO APPROXIMATIONS (I.E. PASS 'SMOOTHLY' FROM NO-LOCALIZATION TO A-LOT-OF-LOCALIZATION)

ϕ^4 theory in 5D

We have the Lagrangian

$$\mathcal{L}_\phi = \frac{1}{2}\partial^M \phi(x,y)\partial_M \phi(x,y) - V^{(5)}(\phi)$$

where the potential is:

$$V^{(5)}(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2$$

Now we look for a y-dependent vev, $\phi_c(y)$.

Euler-Lagrange: $-\phi_c'' + \frac{\partial V^{(5)}}{\partial \phi_c} = 0$

- Trivial solutions (constants):

$$\phi_c = 0 \qquad\qquad \phi_c = \pm\frac{m^2}{\lambda}$$

-Non Trivial solutions:

Euler-Lagrange can be integrated to:

$$\frac{1}{2}\phi_c^{'\,2} - V(\phi_c) = C = \text{const.}$$

- When $C = 0$, the non-trivial solution is

$$\phi_c^{\text{kink}}(y; y_0) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}(y - y_0)\right]$$

This is NOT a periodic solution, i.e. implies infinite extra-dimension.

- When $C \neq 0$, there exist periodic non-trivial solutions:

$$\boxed{\phi_c(y) = \pm \frac{m}{\sqrt{\lambda}} \sqrt{\frac{2k^2}{k^2+1}}\ \mathrm{sn}\left(\frac{m(y-y_0)}{\sqrt{k^2+1}},\ k^2\right)}$$

where

$$k^2 = \frac{m^2 - 2\sqrt{|\lambda C|}}{m^2 + 2\sqrt{|\lambda C|}}$$

and $\mathrm{sn}(x, k^2)$ is the Jacobi elliptic-sine of modulus k, with $0 < k < 1$.

The period of $\mathrm{sn}(x, k^2)$ is $4K$ where

$$K(k^2) = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2 \theta)} = F\left(\frac{\pi}{2}, k\right)$$

is the complete elliptic integral of the first kind.

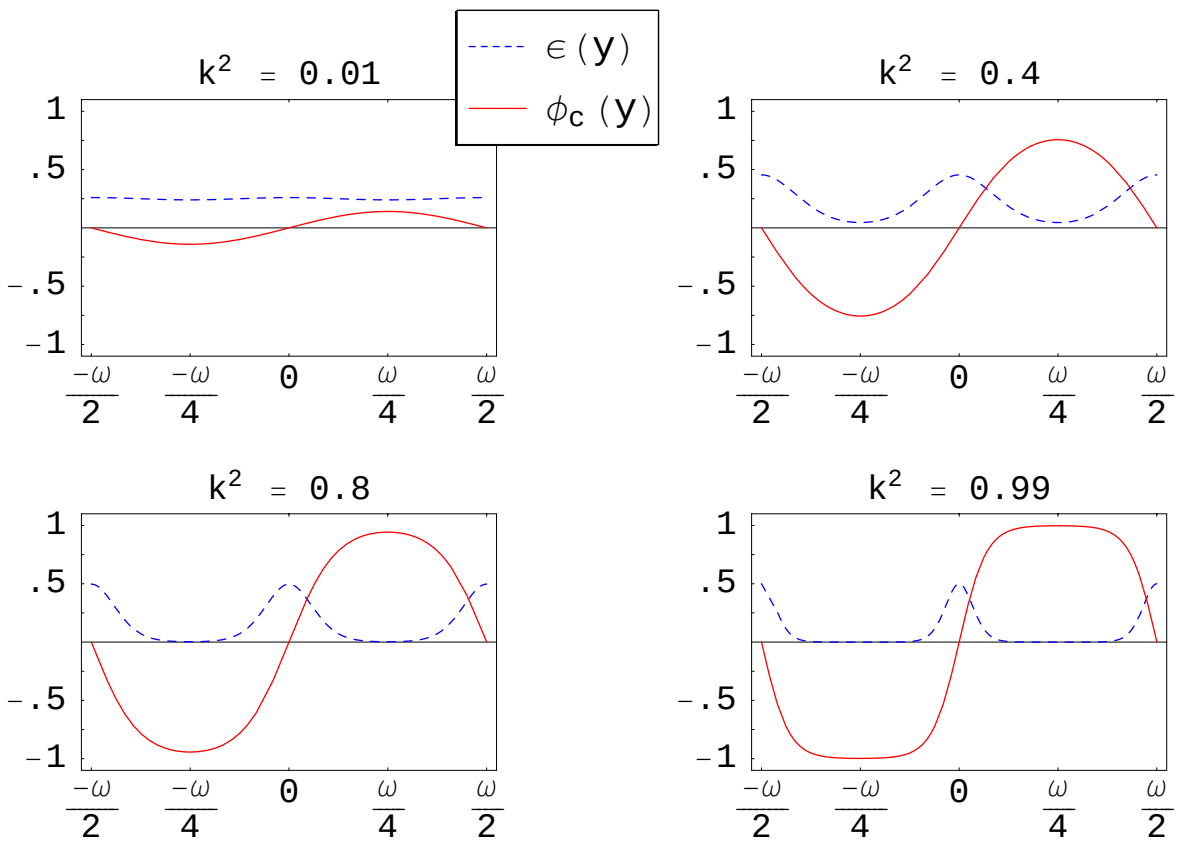


Figure 1: Profiles of $\phi_c(y)$ (in units of $m/\sqrt{\lambda}$) and the "energy density" $\epsilon(y)$ (in units of m^4/λ) for $k^2 = 0.01,\ 0.4,\ 0.8$ and 0.99 (or $\omega = 1.01\omega_0,\ 1.34\omega_0,\ 1.93\omega_0$ and $3.32\omega_0$, with $\omega_0 = 2\pi/m$).

n	ϕ_n	$\frac{M_n^2}{m^2}$	$\phi_n\vert_{k^2=1}$	$\frac{M_n^2}{m^2}\Big\vert_{k^2=1}$	$\frac{M_n^2}{m^2}\Big\vert_{k^2=0}$	Period
0^{++}	$\mathrm{sn}^2(s,k^2) - \frac{1+k^2+\sqrt{1-k^2+k^4}}{3k^2}$	$1-\sqrt{1+3(\frac{1-k^2}{1+k^2})^2}$	$\tanh^2(s)-1$	0	-1	$2K(k^2)$
0^{+-}	$\mathrm{cn}(s,k^2)\ \mathrm{dn}(s,k^2)$	0	$\mathrm{sech}^2(s)$	0	0	$4K(k^2)$
1^{-+}	$\mathrm{sn}(s,k^2)\ \mathrm{dn}(s,k^2)$	$\frac{3k^2}{1+k^2}$	$\mathrm{sech}(s)\ \tanh(s)$	$\frac{3}{2}$	0	$4K(k^2)$
1^{--}	$\mathrm{sn}(s,k^2)\ \mathrm{cn}(s,k^2)$	$\frac{3}{1+k^2}$	$\mathrm{sech}(s)\ \tanh(s)$	$\frac{3}{2}$	3	$2K(k^2)$
2^{++}	$\mathrm{sn}^2(s,k^2) - \frac{1+k^2-\sqrt{1-k^2+k^4}}{3k^2}$	$1+\sqrt{1+3(\frac{1-k^2}{1+k^2})^2}$	$\tanh^2(s)-\frac{1}{3}$	2	3	$2K(k^2)$

Table 1: First five unnormalized scalar eigenfunctions and the resulting mass spectrum. The parities are defined with respect to $s=0$ and $s=K(k^2)$ (and $2K(k^2)$ is the period of $\mathrm{sn}^2(s,k^2)$)

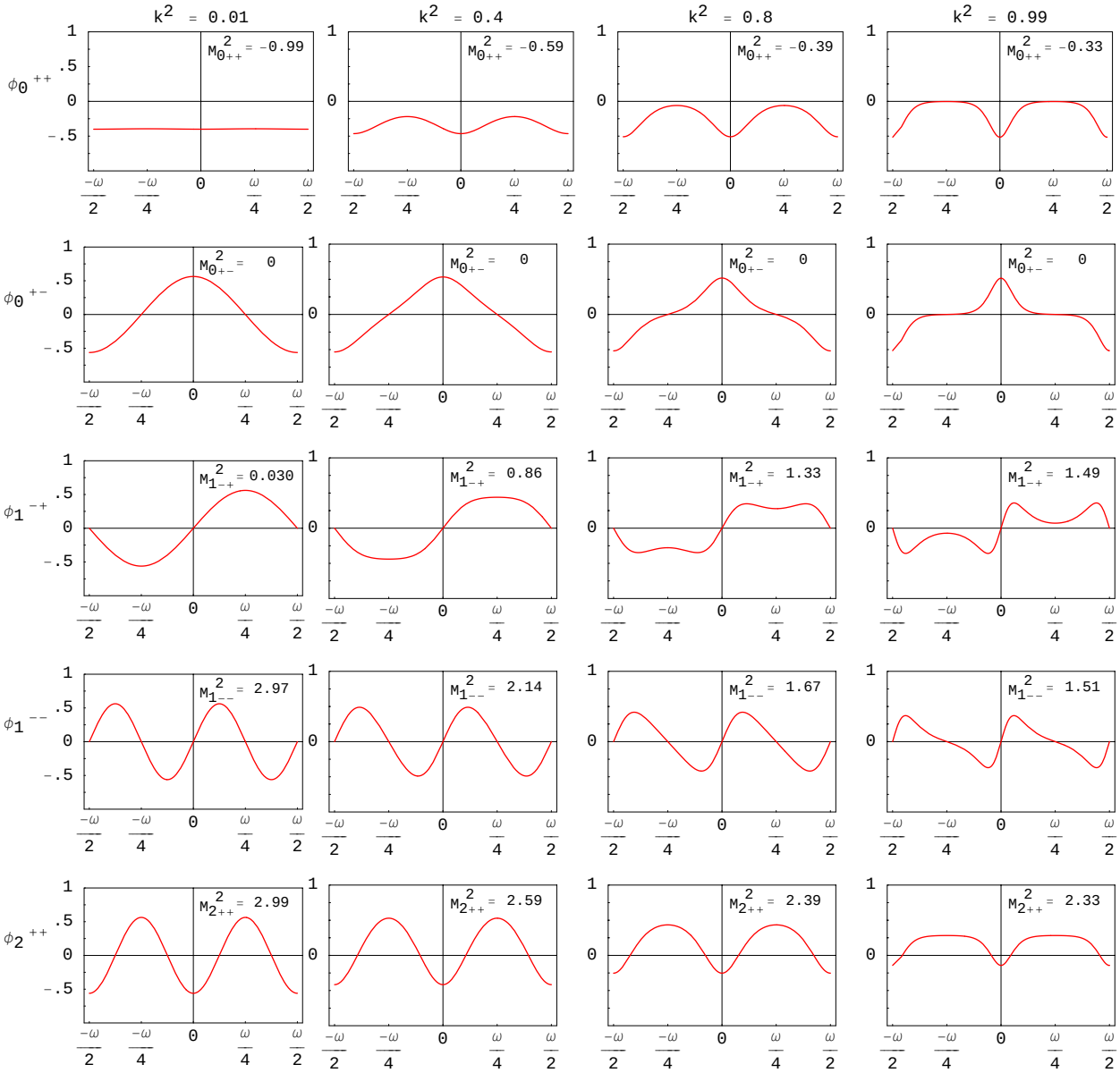


Figure 1: Profile of the first scalar modes for $k^2 = .01,\ .4,\ .8$ and $.99$ (or equivalently $\omega = 1.01,\ 1.34,\ 1.93$ and 3.32 in units of $\omega_0 = 2\pi/m$). We have chosen $y_0 = 0$, for $y_0 \neq 0$ one should replace y by $y + y_0$.

Scalar Perturbations

$$\phi(x,y) \to \phi_c(y) + \phi(x,y)$$

We separate variables and find the y-component equation:

$$\begin{aligned} -\phi_n''(y) + \left(\frac{\partial^2 V^{(5)}}{\partial \phi^2}\Big|_{\phi_c} - M_n^2 \right) \ \phi_n(y) &= 0 \\ -\phi_n''(y) + \left[3\lambda \phi_c^2(y) - m^2 - M_n^2 \right] \ \phi_n(y) &= 0 \end{aligned}$$

By a suitable rescaling, $s = \frac{m}{\sqrt{k^2+1}}\ y$, this becomes the Lamé equation

$$\boxed{\phi_n''(s) + \left[h - n(n+1)k^2 \mathrm{sn}^2(s,\ k^2) \right] \ \phi_n(s) = 0}$$

where $n = 2$ and $h = (k^2+1)(1 + \frac{M_n^2}{m^2})$.

Fermion Localization

We start with a 5D massless Fermion Lagrangian:

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(x,y)\left[i\Gamma^M\partial_M - f\phi(x,y)\right]\psi(x,y) \\ &\quad + \mathcal{L}_\phi\end{aligned}$$

REMARKS

- A Z_2 symmetry (parity under $y \to -y$) forbids a fermion mass term.
- Massless 5D fermions are not chiral since $\Gamma^5 = i\gamma_5$.
- The scalar field ϕ acquires a nontrivial vev $\phi_c(y)$ ($\phi(x,y)$ is odd under Z_2 and we avoid the instability).

We decompose $\psi(x,y)$ in chiral components:

$$\psi \;=\; \psi_{\rm L} + \psi_{\rm R}$$

REMARK:

- Our parity choice for ψ is such that:
  $\psi_{\rm L}(y) = -\psi_{\rm L}(-y)$
  $\psi_{\rm R}(y) = \psi_{\rm R}(-y)$

The coupled Weyl equations are:

$$i\partial\!\!\!/\; \psi_{\rm L}(x,y) - (\partial_5 + f\phi_c(y))\psi_{\rm R}(x,y) = 0$$
$$i\partial\!\!\!/\; \psi_{\rm R}(x,y) + (\partial_5 - f\phi_c(y))\psi_{\rm L}(x,y) = 0\,.$$

We separate variables for each chiral component

$$\psi_{\rm L}(x,y) \;=\; \psi_x^{\rm L}(x)\;\psi_y^{\rm L}(y)$$

$$\psi_{\rm R}(x,y) \quad = \quad \psi_x^{\rm R}(x)\ \psi_y^{\rm R}(y)\,,$$

which gives for the x-components

$$i\partial\!\!\!/\ \psi_x^{\rm L} - m_n \psi_x^{\rm R} = 0$$
$$i\partial\!\!\!/\ \psi_x^{\rm R} - m_n \psi_x^{\rm L} = 0$$

and for the y-components

$$(\partial_y + f\phi_c(y))\psi_y^{\rm R} = m_n \psi_y^{\rm L}$$
$$(-\partial_y + f\phi_c(y))\psi_y^{\rm L} = m_n \psi_y^{\rm R} \quad .$$

m_n is the constant of separation.

- **Zero mode**

If $m_n = 0$, these become

$$(\partial_y + f\phi_c(y))\psi_y^{\rm R} = 0$$
$$(-\partial_y + f\phi_c(y))\psi_y^{\rm L} = 0 \quad .$$

and can be integrated

$$\psi_0^{\mathrm{R}}(y) = k_1 \exp\left\{-f \int_{y_0}^{y} d\tilde{y}\ \phi_c(\tilde{y})\right\}$$

and

$$\psi_0^{\mathrm{L}}(y) = k_2 \exp\left\{f \int_{y_0}^{y} d\tilde{y}\ \phi_c(\tilde{y})\right\}$$

and because of the parity imposition, we have $k_2 = 0$, hence $\psi_0^{\mathrm{L}}(y) = 0$

When $\phi_c = \pm\frac{m}{\sqrt{\lambda}}\sqrt{\frac{2k^2}{k^2+1}}\ \mathrm{sn}\left(\frac{m(y-y_0)}{\sqrt{k^2+1}},\ k^2\right)$, the solution for $\psi_0^{\mathrm{R}}(y)$ is amazingly simple:

$$\boxed{\psi_0^{\mathrm{R}}(y) \propto \left[\mathrm{dn}\left(\frac{m}{\sqrt{k^2+1}}y,\ k^2\right) + k\ \mathrm{cn}\left(\frac{m}{\sqrt{k^2+1}}y,\ k^2\right)\right]^{\delta}}$$

where $\delta = f\sqrt{2/\lambda}$.

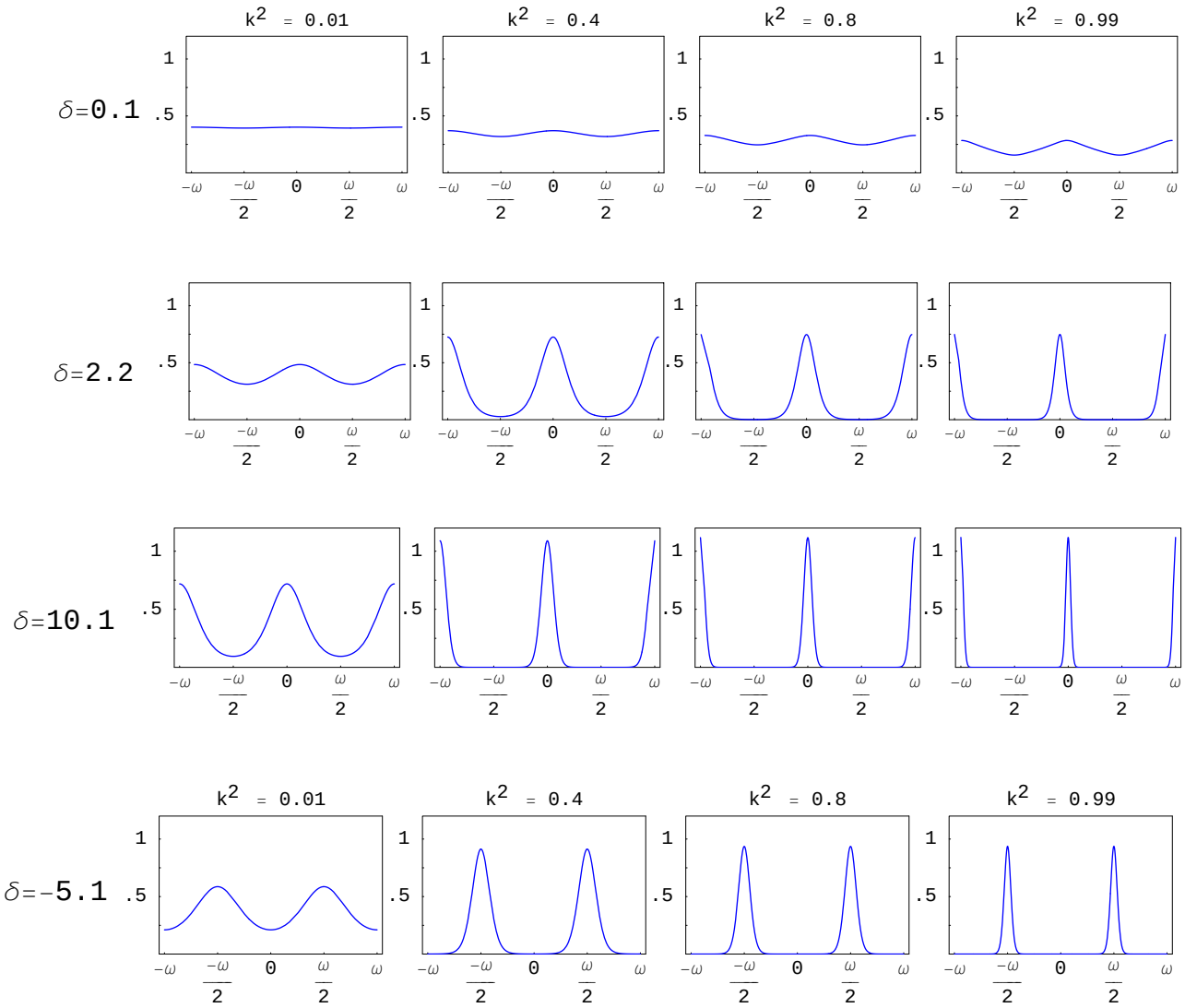


Figure 2: Normalized zero mode ψ_0^{R} for $\omega = 1.01\omega_0,\ 1.34\omega_0,\ 1.93\omega_0$ and $3.32\omega_0$, with $\omega_0 = 2\pi/m$ and with $\delta = 0.1,\ 2.2,\ 10.1$ and -5.1

- General Fermion Spectrum

  We had:

$$
\begin{aligned}
(\partial_y + f\phi_c(y))\psi_y^{\mathrm{R}} &= m_n \psi_y^{\mathrm{L}} \\
(-\partial_y + f\phi_c(y))\psi_y^{\mathrm{L}} &= m_n \psi_y^{\mathrm{R}} \quad .
\end{aligned}
$$

  We decouple them:

$$
\begin{aligned}
\left[-\partial_y^2 + (f\phi_c)^2 - f\phi_c' - m_n^2\right] \psi_y^{\mathrm{R}} &= 0 \\
\left[-\partial_y^2 + (f\phi_c)^2 + f\phi_c' - m_n^2\right] \psi_y^{\mathrm{L}} &= 0
\end{aligned}
$$

Inserting our ϕ_c, and introducing $s = \frac{m}{\sqrt{k^2+1}}\, y$, we have

$$\left[\frac{d^2}{ds^2} - (\delta\ k)^2 \mathrm{sn}^2 \pm (\delta\ k)\ \mathrm{cn}\ \mathrm{dn} + (k^2+1)\frac{m_n^2}{m^2}\right]\psi_s^{\mathrm{R/L}} = 0$$

REMARKS:

- Periodic solutions will be of period ω and 2ω:

$$\begin{aligned} \psi_\omega^R(y) &= \psi_\omega^R(y+\omega) \\ \psi_{2\omega}^R(y) &= -\psi_{2\omega}^R(y+\omega) \end{aligned}$$

- Antiperiodic solutions (2ω) are allowed: Sherk-Schwarz compactification with Twist of -1

- Equivalently, we could think of an $S1/(Z2 \times Z2')$ compactification of period 2ω.

$\rightarrow$ Periodic solutions correspond to $(+,+)$

$\rightarrow$ AntiPeriodic solutions to $(+,-)$

Two limits are known:

a) $(k \to 0)$, the equations simplify to

$$\left(\frac{d^2}{ds^2} + \frac{m_n^2}{m^2}\right)\psi_s^{\mathrm{R/L}} = 0\,,$$

with $s = my$.

b) $(k \to 1)$, the equations become

$$\left(-\frac{d^2}{ds^2} + \delta^2 \tanh^2 s \mp \delta\ \mathrm{sech}^2 s\ - 2\frac{m_n^2}{m^2}\right)\psi_s^{\mathrm{R/L}} = 0$$

for $s = my/\sqrt{2}$.

$\to$Georgi,Grant,Hailu [hep-ph/0007350]

SOLUTIONS for ψ^R:

$$\psi^R_{2n}(y) \;\propto\; \psi_0(y) \;\; CI^{2n}_{\omega}\left[\,\mathrm{am}\left(\frac{m}{2}\frac{(1+k)}{\sqrt{k^2+1}}y,\,\frac{4k}{(1+k)^2}\right)\right]$$

$$\psi^R_{2n+1}(y) \;\propto\; \psi_0(y) \;\; CI^{2n+1}_{2\omega}\left[\,\mathrm{am}\left(\frac{m}{2}\frac{(1+k)}{\sqrt{k^2+1}}y,\,\frac{4k}{(1+k)^2}\right)\right]$$

where

$$\psi_0(y) = \left[\mathrm{dn}\left(\frac{m}{\sqrt{k^2+1}}\;y,\;k^2\right) + k\;\mathrm{cn}\left(\frac{m}{\sqrt{k^2+1}}\;y,\;k^2\right)\right]^{\delta} \tag{1}$$

$$CI^{2n}_{\pi}(t) \;=\; \sum_{r=0}^{\infty} A^{2n}_{2r}\cos 2rt$$

$$CI^{2n+1}_{2\pi}(t) \;=\; \sum_{r=0}^{\infty} C^{2n+1}_{2r+1}\cos(2r+1)t$$

where $n = 1, 2, 3, \ldots$.

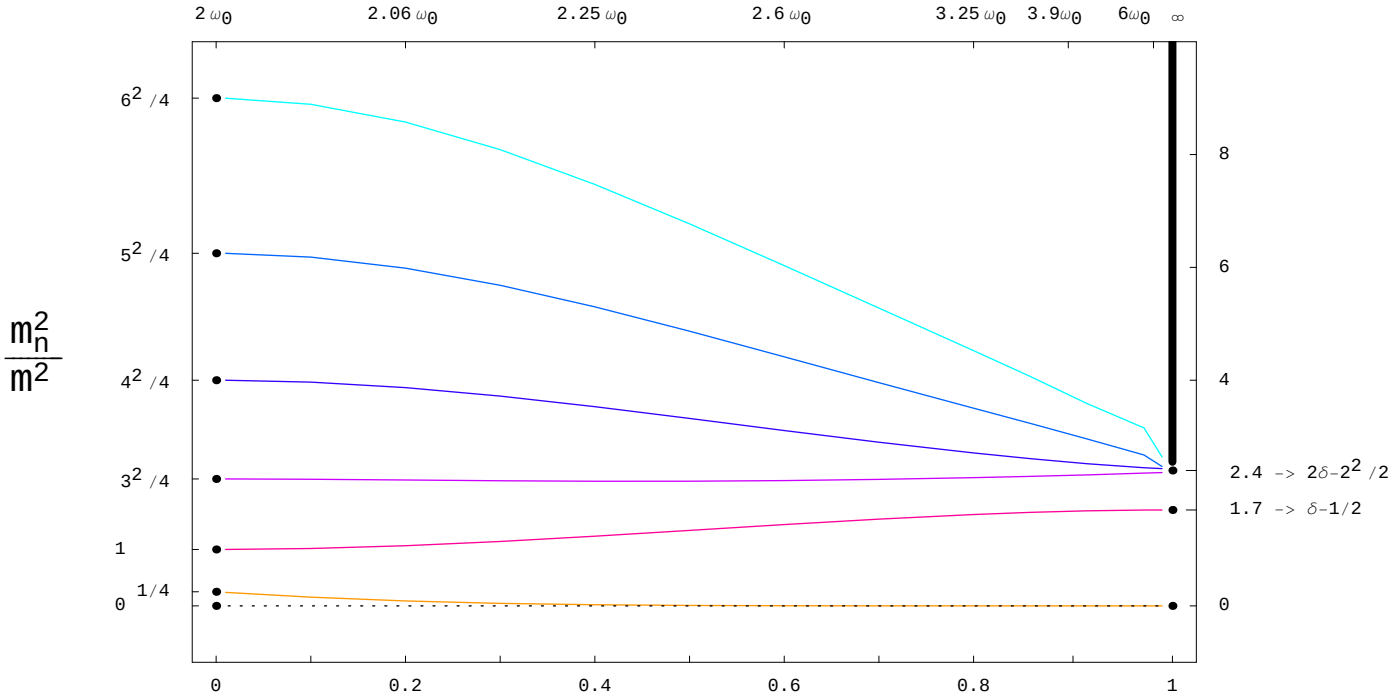


Figure 3: Spectrum of the first 7 modes with respect to the size of the extra dimension with $\delta = 2.2$

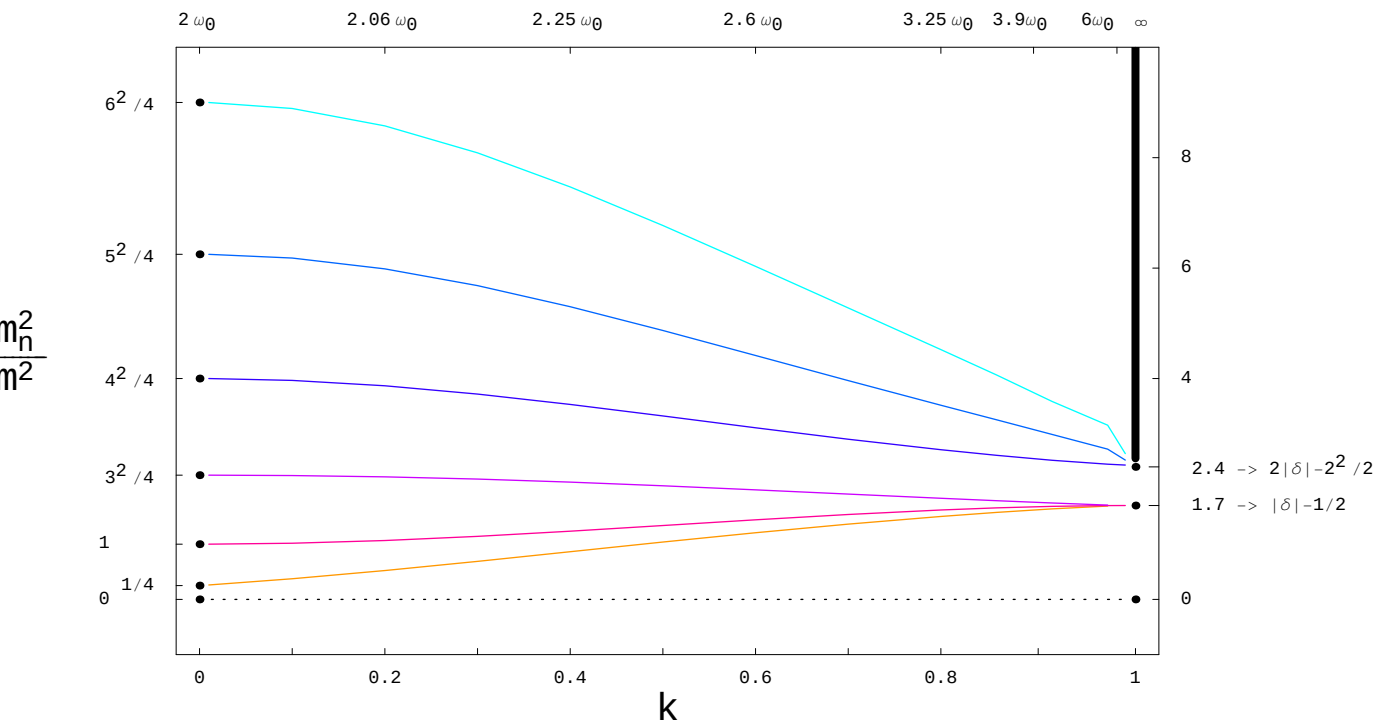


Figure 4: Same with $\delta = -2.2$.

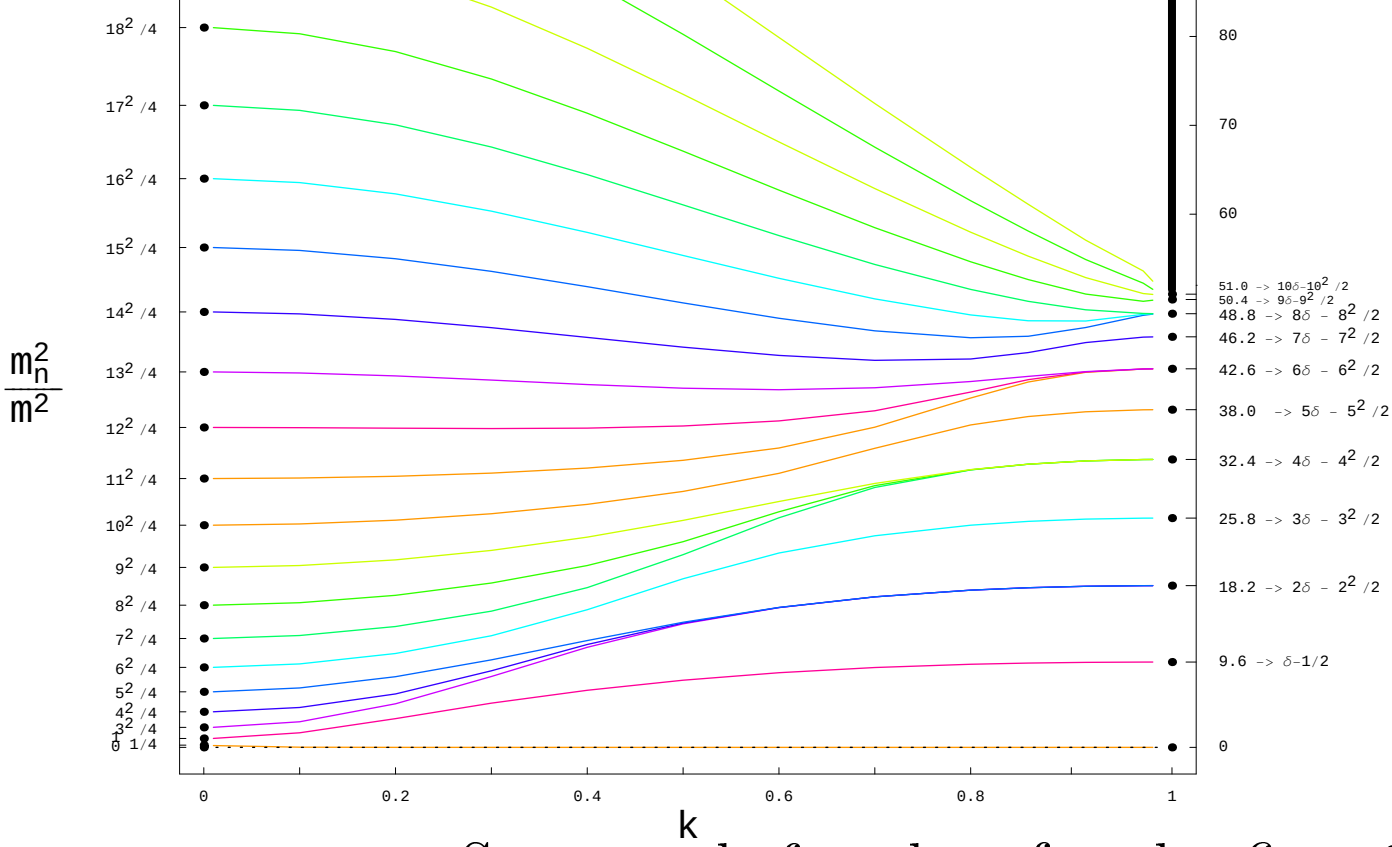


Figure 5: Same as before but for the first 22 modes and with $\delta = 10.1$

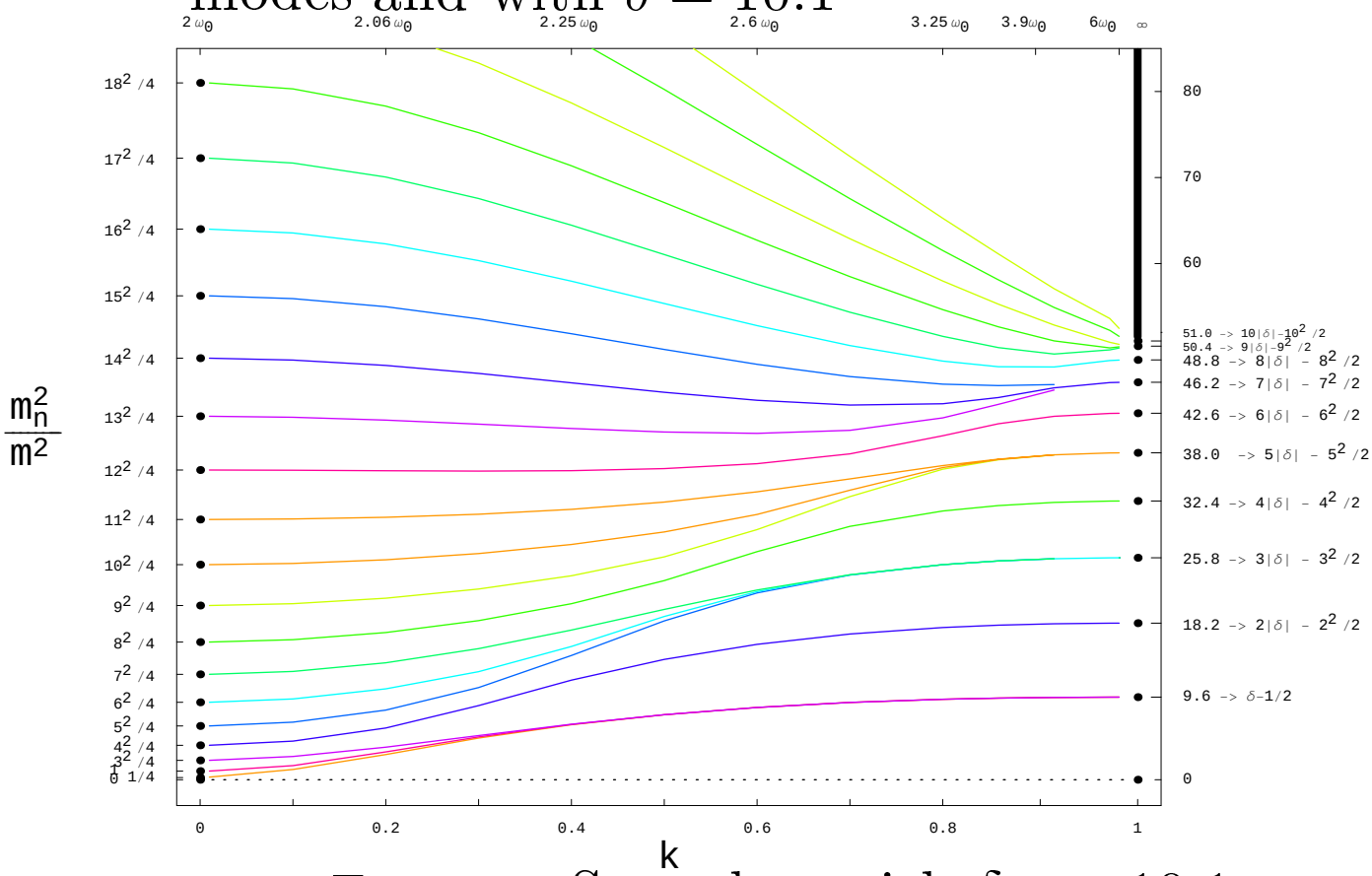


Figure 6: Same but with $\delta = -10.1$.

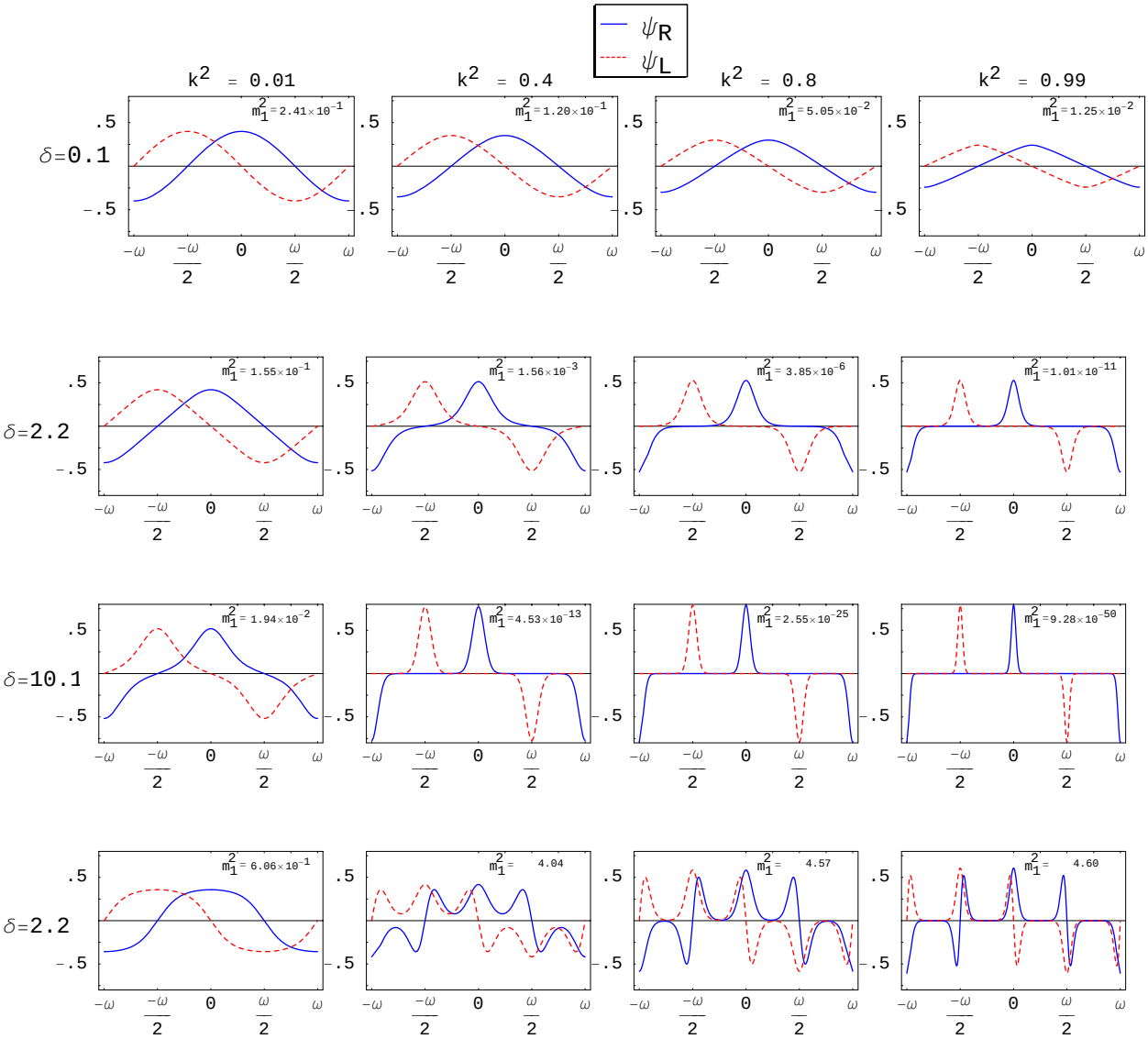


Figure 7: Profile of $\psi_1^{\mathrm{R/L}}$, of period 2ω (for $\omega = 1.01\omega_0$, $1.34\omega_0$, $1.93\omega_0$ and $3.32\omega_0$) and for $\delta = 0.1$, 2.2, 10.1 and -5.1. The mass eigenvalues are given in units of m^2.

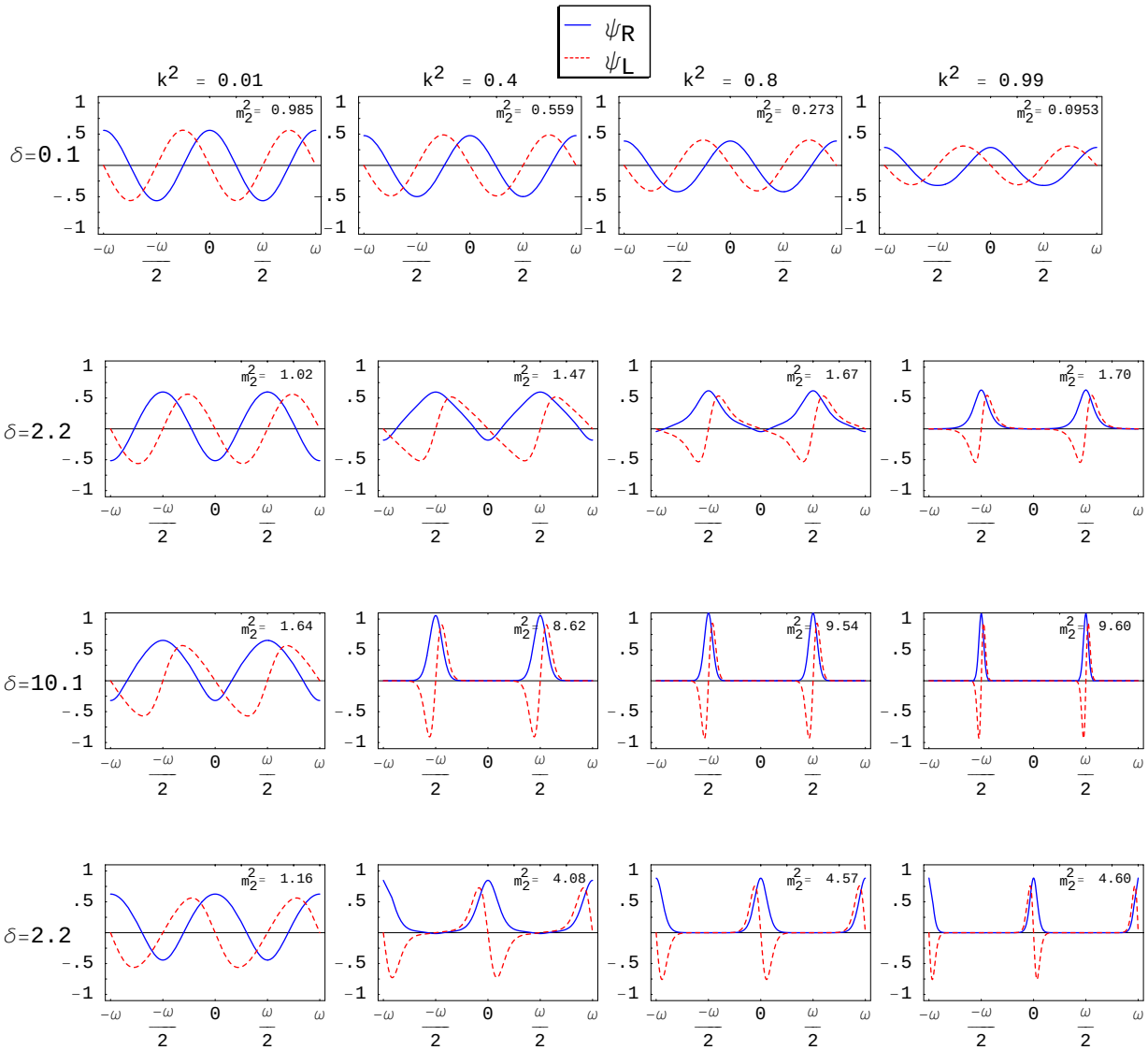


Figure 8: Profile of $\psi_2^{\mathrm{R/L}}$, of period ω (for $\omega = 1.01\omega_0,\ 1.34\omega_0,\ 1.93\omega_0$ and $3.32\omega_0$) and with $\delta = 0.1,\ 2.2$ and 10.1. The mass eigenvalues are given in units of m^2.

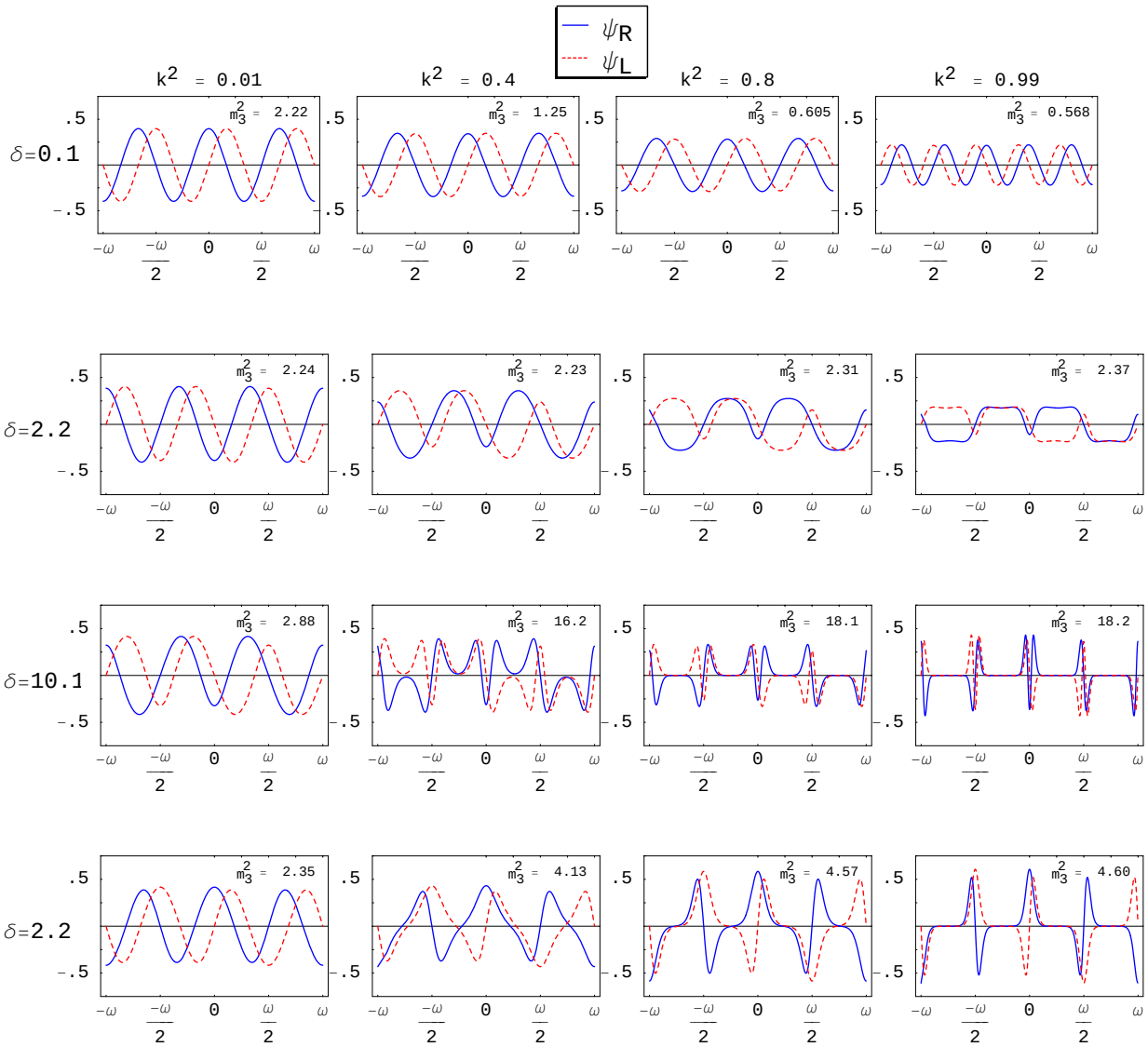


Figure 9: Profile of $\psi_3^{\mathrm{R/L}}$, of period 2ω (for $\omega = 1.01\omega_0,\ 1.34\omega_0,\ 1.93\omega_0$ and $3.32\omega_0$) and with $\delta = 0.1,\ 2.2$ and 10.1. The mass eigenvalues are given in units of m^2.

Outlook

- Solved ϕ^4 with a periodic vev.
- Instability solved by imposing Z_2 symmetry
- Found exact solution for localized mode ψ_0
- Found numerically the fermionic spectrum for $(++)$ and $(+-)$ modes.

Make use of the very light $(+-)$ modes?