P445: QFT 3 : SOLUTIONS SET 1 19/4/06

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1. Action invariant $\Rightarrow$ $\delta\mathcal{L}$ invariant up to surface term.

(a)

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\,\delta(\partial_\mu\Phi_i) + \frac{\partial\mathcal{L}}{\partial\Phi_i}\,\delta\Phi_i \qquad (*)$$

The transformation $\delta\Phi_i$ is generated by symmetry, i.e.

$$\delta\Phi_i = i\varepsilon^a T^a_{ij}\Phi_j \quad \leftarrow \text{keep it generic for group } G$$

$$\delta\,\partial_\mu\Phi_i = i\varepsilon^a T^a_{ij}\,\partial_\mu\Phi_j$$

and by Equation of motion

$$\frac{\partial\mathcal{L}}{\partial\Phi_i} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}$$

$$\Rightarrow (*) = i\varepsilon^a\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\,T^a_{ij}\Phi_j\right] = 0$$

Since ε^a is global, $J^{\mu a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\,T^a_{ij}\Phi_j$

is a conserved current.

QFT3 Soln 1

Specialising to $U(1)$: $\delta\Phi = i\varepsilon\Phi$ $\quad \delta\Phi^* = -i\varepsilon\Phi^*$ $\quad T^a = \mathbb{1}$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\Phi^*$$

$$= \partial_\mu\Phi^*\Phi - (\partial_\mu\Phi)\Phi^*$$

(b) Let $\Phi \equiv \frac{1}{\sqrt{2}} e^{i\xi(x)} \rho(x)$

$$J^\mu = \frac{1}{2}\left(-i\,\partial_\mu\xi\, \cancel{e^{-i\xi}}\rho + \cancel{e^{-i\xi}}\partial_\mu\rho\right)\cancel{e^{i\xi}}\rho$$
$$- \frac{1}{2}\left(i\,\partial_\mu\xi\, \cancel{e^{i\xi}}\rho + \cancel{e^{i\xi}}\partial_\mu\rho\right)\cancel{e^{-i\xi}}\rho$$

$$= -\frac{i}{2}\left(\partial_\mu\xi\, \rho^2\right)$$

Break symmetry:

$$\Phi = \frac{1}{\sqrt{2}} e^{i\xi/v}(v+\rho)$$

NG mode, properly normalised $\Rightarrow$ massless and kinetic term has coeff $\frac{1}{2}$.

QFT 3: Solutions 1

$$\mathcal{L} = \partial_\mu \Phi \, \partial^\mu \Phi^* - V(\Phi^* \Phi)$$

$$= \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\partial_\mu \xi)^2 \left(1 + \frac{2\rho}{v} + \frac{\rho^2}{v^2}\right) - V(\rho^2)$$

- kinetic term has $\frac{1}{2} \Rightarrow$ properly normalised
- no mass term for ξ since $V(\Phi^* \Phi) \to V(\rho)$

(c) Noether current is

$$j^\mu = -\frac{i}{2} v \left(\partial_\mu \xi \left(1 + \rho/v\right)^2\right)$$

So if you include the ~~interaction~~ kinetic term for ξ

$$\mathcal{L}_{int} \sim \frac{1}{2} \partial_\mu \xi \left[\partial_\mu \xi \left(1 + \rho/v\right)^2\right]$$

$$\sim -\frac{1}{2} \xi \, \partial_\mu \left(\downarrow \right) \sim \frac{1}{v} \xi \, \partial_\mu j^\mu$$

(by parts)

QFT 3: Solutions 1

Have symmetry gp G with generators T^a.
Assume G gets broken, s.t. we now have

$$T'^A \vec{\phi}_0 = 0 \quad \leftarrow$$ unbroken generators
$\rightarrow$ vacuum still invariant here

$$T''^z \vec{\phi}_0 \neq 0 \quad \leftarrow$$ broken generators

$\underline{\phi}_0$ is the equivalent of v for larger groups, in the appropriate representation.

<u>Pick</u> $$\phi(x) = \exp\left(i\,\theta^z(x) T''^z\right)\left(\vec{\phi}_0 + \vec{\sigma}(x)\right)$$

with $$\vec{\sigma}(x)^T T^{H z} \vec{\phi}_0 = 0, \qquad (**)$$

i.e. $\sigma(x)$ is only non-zero in unbroken directions

QFT3 : Solutions 1

$$\partial_\mu \phi = i \partial_\mu \theta^2 T^{12} \exp(i \theta^2 T^{12}) (\phi_0 + \cancel{\phi})$$

$= 0$ by (**)

$$+ \exp(i \theta^2 T^{12}) \partial_\mu \sigma$$

So $$\partial_\mu \phi^\dagger = \partial_\mu \sigma^\dagger \exp(-i T^{12\dagger} \theta^{12\dagger})$$

$$- i (\phi_0 + \cancel{\phi})^\dagger \exp(-i T^{12\dagger} \theta^{2\dagger}) T^{12\dagger} \partial_\mu \theta^{12}$$

Expanding kinetic term

$$\partial_\mu \phi^\dagger \partial_\mu \phi = \partial_\mu \sigma^\dagger \partial^\mu \sigma + |\phi_0|^2 \partial_\mu \theta^{2\dagger} \partial^\mu \theta^2$$

$$+ \text{ cross terms (interactions)}$$

The potential $V(\phi^\dagger \phi) \rightarrow V(\sigma^\dagger \sigma)$

$\Rightarrow$ σ's are massive

θ's are massless $\leftarrow$ NG bosons, one for each broken generator.

QFT3: Solutions 1

2. (a) $U(1) \times U(1)$

Covariant derivative:

$$D_\mu \phi = (\partial_\mu - ig A_\mu + ig' B_\mu)\phi$$

charges $(+1, -1)$ under $U(1) \times U(1)$

So $$\mathcal{L}_{kin} = (D_\mu \phi)^\dagger (D^\mu \phi) = \partial_\mu \phi' \partial^\mu \phi' + \text{interactions}$$

$$+ \frac{1}{2}(A_\mu \;\; B_\mu) v^2 \underbrace{\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix}$$

Diagonalise $\underline{\underline{M}}$ to get mass spectrum.

$$\Rightarrow \quad m^2 = 0, \; 2v^2(g^2 + g'^2)$$

with eigenvectors

$$\frac{gA^\mu + g'B^\mu}{\sqrt{g^2 + g'^2}} \qquad \frac{g'A^\mu - gB^\mu}{\sqrt{g^2 + g'^2}}$$

QFT 3: Solutions 1

SU(2):

$$D_\mu \phi = \left(\partial_\mu - i g A_\mu^a T^a\right) \phi$$

ϕ is in adjoint $\Rightarrow$ $(T^a)_{ij} = i\varepsilon_{iaj}$

Pick $\langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

Mass matrix from

$$\mathcal{L}_m \sim \frac{1}{2} g^2 \begin{pmatrix} 0 & 0 & v \end{pmatrix} 2 \left(A^{\mu a} T^a T^b A_\mu^b \right) \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

gives:

$$M = 2v^2 g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ get 2 massive states

$m^2 = 2v^2 g$ (charge ± 1)

one massless $m^2 = 0$. (uncharged)

In SM, have additional massive state, uncharged

$\Rightarrow$ need more complicated group