Problem 1.

To solve the first part of the problem, one must require that the moving car fits into the garage, namely

$$L_C\sqrt{1 - v^2/c^2} = L_G \tag{1}$$

or, equivalently

$$v = c\sqrt{1 - L_G^2/L_C^2} \tag{2}$$

In the reference frame K, where the garage is at rest, the opening and closure of the doors occurs are simultaneous. In the reference frame K', where the car is at rest, the two processes occur at different times, namely:

$$\begin{aligned} \Delta t' &= (-v/c^2\Delta x)/\sqrt{1 - v^2/c^2} \\ &= -\frac{L_C}{c}\sqrt{1 - \frac{L_G^2}{L_C^2}} \end{aligned} \tag{3}$$

The front door opens first, the car advances, and then, after a time $\Delta t'$, the rear door closes.

The length of the moving garage in the reference frome K' is equal to

$$L'_G = L_G\sqrt{1 - v^2/c^2} = L_G^2/L_C \tag{4}$$

Observe that $\Delta t'$ is precisely the time that the car needs to advance in order to avoid hitting the doors, namely

$$\Delta t' = -\frac{L_C - L'_G}{v} \tag{5}$$

Problem 2.

In the system K'

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2} \tag{6}$$

On the other hand, in the sytem K"

$$u''_x = \frac{u'_x - v'}{1 - v'u'_x/c^2} \tag{7}$$

Replacing Eq. (6) into Eq. (7), we get

$$u''_x = \frac{u_x - v''}{1 - v''u_x/c^2} \qquad \text{with} \qquad v'' = \frac{v + v'}{1 + vv'/c^2} \tag{8}$$

Notice that v'' is the speed of K" with respect to K, as can be easily proved by transforming the velocity v' from the system K' into the reference frame K

Observe that the expression is symmetric under the change of $v \leftrightarrow v'$. Then, $S(v)S(v') = S(v')S(v)$.

In general, the two Lorentz transformation do not commute, as can be easily shown by trial and error, or by simply considering Lorentz transformation as "rotations" in space-time (consider infinitesimal transformations in perpendicular directions. The generators do not commute).

Problem 3.

There are two ways of solving the first part of the problem. First we notice that the four-acceleration vector takes the form

$$w^\mu = \frac{\gamma}{c}\frac{d}{dt}\left(\gamma, \frac{\vec{v}}{c}\gamma\right) \tag{9}$$

We can do a transformation to the rest frame of the particle to compute the constant acceleration w, which, assuming a motion along the x-axis is equal to $w/c^2 = w'^1$ (Observe that w'^0 vanishes in the system in which the particle is at rest, since it is proportional to the velocity of the particle.

$$w'^1 = (w^1 - v/cw^0)\gamma = \frac{\gamma^2}{c}\left(\gamma\frac{d(v/c)}{dt}\right) \tag{10}$$

Namely,

$$w = \gamma^3\frac{dv}{dt} = \frac{d(v\gamma)}{dt} \tag{11}$$

The other way is to compute $w^\mu w_\mu = -w^2/c^4$. Taking the above expresion, and taking into account the relation between $d(v\gamma)$ and dv given in Eq. (11), we get

$$w^\mu = \frac{\gamma^4}{c}\,(v/c, 1, 0, 0)\,\frac{d(v/c)}{dt} \tag{12}$$

From where

$$-\frac{w^2}{c^4} = \frac{1}{c^2\,(1 - v^2/c^2)}\left[\frac{v^2/c^2 - 1}{(1 - v^2/c^2)^3}\right]\left(\frac{d(v/c)}{dt}\right)^2 \tag{13}$$

and we recover Eq. (11).

Assuming that the particle is at rest at $t = 0$, we obtain

$$v\gamma = wt \tag{14}$$

or, equivalently

$$v(t) = \frac{dx}{dt} = \frac{w\;t}{\sqrt{1 + w^2t^2/c^2}} \tag{15}$$

Integrating over t, and assuming that $x(t = 0) = 0$, we get

$$x = \frac{c^2}{w}\left[\left(1 + w^2t^2/c^2\right)^{1/2} - 1\right] \tag{16}$$

For small tw/c, we get $x = wt^2/2$.

The proper time τ interval is given by

$$\Delta\tau = \int_0^t \gamma^{-1} dt = \int_0^t \left(1 - \frac{w^2t^2/c^2}{1 + w^2t^2/c^2}\right)^{1/2} \tag{17}$$

It is trivial to integrate this expression. We get

$$\Delta\tau = \frac{c}{w}\sinh^{-1}\left(\frac{wt}{c}\right) \tag{18}$$

Problem 4.

This is just an attempt of solving the problem numerically. I haven't been too careful in inserting the numbers, so a further check would be necessary.

Taking the last part of the solution of problem 3, and defining $\Delta\tau = 5$ years, $w = g = 9.8 m/s^2$, the time on earth is given by (Observe that we are using the property that the difference in time measured in both systems do not depend on the direction of the velocity).

$$\Delta t = 4\frac{c}{g}\sinh\left(\frac{g\Delta\tau}{c}\right) \tag{19}$$

The value of $g\Delta\tau/c = 5.151$, and $\sin h(g\Delta\tau/c) = 86.29$. So, in earth 355.57 years have passed.

The travelling twin travelled a distance

$$\Delta x = 2\frac{c^2}{g}\left[\cosh\left(\frac{g\Delta\tau}{c}\right) - 1\right] \tag{20}$$

before starting to come back, what gives a distance of $1.56\ 10^{18}$ meters.

Observe that the traveller spends most of the time travelling at velocities close to the speed of light. Indeed, after the first year he is already travelling at a speed larger than 0.7 c. After the second year he starts travelling at velocities larger than 0.9 c. After the third year, larger than 0.95 c, etc...