Problem 1

Let's take spherical coordinates, with the z-axis pointing towards the direction of $\vec{\beta}$ and the x-axis pointing towards the direction of the acceleration. Then, we obtain

$$\frac{dP}{d\Omega} = \frac{q^2\dot{\beta}^2}{4\pi c}\left[\frac{1}{(1-\beta\cos\theta)^3} - \frac{\cos^2\phi\sin^2\theta}{(1-\beta\cos\theta)^5\gamma^2}\right] \tag{1}$$

Let's integrate the above expression over θ. For that one has to complete the squares and express

$$\beta^2\cos^2\theta = \left[(1-\beta\cos\theta)^2 - 2(1-\beta\cos\theta) + 1\right] \tag{2}$$

With this trick, it is trivial to demonstrate that

$$\int d\cos\theta \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = -\frac{4}{3}\gamma^6 \tag{3}$$

Moreover,

$$\int d\cos\theta \frac{1}{(1-\beta\cos\theta)^3} = -2\gamma^4 \tag{4}$$

Therefore,

$$\frac{dP}{d\phi} = -\int_0^\pi d\cos\theta \frac{dP}{d\Omega} = \frac{q^2\gamma^4\dot{\beta}^2}{4\pi c}\left[2 - \cos^2\phi\frac{4}{3}\right] \tag{5}$$

The emission of energy is maximal along the plane y-z, perpendicular to the direction of the acceleration, and is minimized along the plane x-z.

Integrating the above expression over $d\phi$, one obtains

$$P(t') = \frac{2q^2\dot{\beta}^2\gamma^4}{3c} \tag{6}$$

Problem 2

We start with the non-relativistic approximation

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}|\hat{n} \text{ x } (\hat{n} \text{ x } \dot{\vec{\beta}}|^2 \tag{7}$$

Here $\vec{r} = \hat{z} a \cos\omega_0 t$, and

$$\dot{\vec{\beta}} = -\hat{z}\frac{a\omega_0^2 \cos\omega_0 t}{c} \tag{8}$$

Hence

$$\frac{dP}{d\Omega} = \frac{q^2 a^2 \omega_0^4}{4\pi c^3} \cos^2\omega_0 t \sin^2\theta \tag{9}$$

where θ is the angle between the $\hat{z}$ and $\hat{n}$. The time average is very simple, since

$$\int_0^T \cos^2\omega_0 t = \frac{T}{2} \tag{10}$$

Then,

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{q^2 a^2 \omega_0^4}{8\pi c^3} \sin^2\theta \tag{11}$$

where the brackets denote time average. Integrating over θ one obtains

$$< P >= \frac{q^2 a^2 \omega_0^4}{3c^3} \tag{12}$$

(b) This time $\vec{r} = R(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t)$, while $\dot{\vec{\beta}} = -\omega_0^2 \vec{r}/c$. Expressing things in spherical coordinates we get

$$|\hat{n} \text{ x } (\hat{n} \text{ x } \dot{\vec{\beta}})|^2 = \frac{\omega_0^4 R^2}{c^2}\left[1 - \sin^2\theta(\cos\phi\cos\omega_0 t + \sin\phi\sin\omega_0 t)\right] \tag{13}$$

Hence,

$$\frac{dP}{d\Omega} = \frac{q^2 \omega_0^4 R^2}{4\pi c^3}\left[1 - \sin^2\theta\cos^2(\omega_0 t - \phi)\right] \tag{14}$$

Using the fact that the ime average $< \cos^2(\omega_0 t - \phi) > = 0.5$, we get

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{q^2 \omega_0^4 R^2}{8\pi c^3}(1 + \cos^2\theta) \tag{15}$$

Observe that radiation is maximal along the z-axis. Finally, integrating over θ an ϕ we get

$$< P >= \frac{2q^2 \omega_0^4 R^2}{3c^3} \tag{16}$$

Problem 3.

a) We know from class that the total radiation power is given by

$$P = \frac{2q^2 \dot{\vec{\beta}}^2 \gamma^4}{3c} \tag{17}$$

The acceleration of the particle is equal to $R\omega_B^2$, where

$$\omega_B = \frac{qB}{\gamma mc} \tag{18}$$

Thus,

$$P = \frac{2q^2\gamma^4 R^2 \omega_B^4}{3c^3} \tag{19}$$

Now, putting $\omega_B = v/R$ and solving for R, we get

$$R = \frac{\gamma cmv}{qB} \tag{20}$$

and using the relation between v/c and γ, we get that

$$R^2 = \frac{m^2c^4}{q^2B^2}(\gamma^2 - 1) \tag{21}$$

Thus,

$$P(t') = \frac{2q^4B^2(\gamma^2 - 1)}{3m^2c^3} \tag{22}$$

b) From point (a) we have that

$$dE_{rad} = -dE = \frac{2q^4B^2(\gamma^2 - 1)}{3m^2c^3} dt' \tag{23}$$

where E is the energy of the particle. But $E = \gamma mc^2$, so $dE = mc^2 d\gamma$. Hence,

$$\int_{\gamma(t=0)}^{\gamma(t)} \frac{d\gamma}{\gamma^2 - 1} = -\frac{2q^4B^2}{3m^3c^5} \int_0^t dt' \tag{24}$$

If $\gamma \gg 1$ during the time interval above, then we can approximate the left hand integrand by $1/\gamma^2$. Thus,

$$t \simeq \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma(t)} - \frac{1}{\gamma(t=0)}\right) \tag{25}$$

c) In the non-relativistic limit, $\gamma \simeq 1$ and then

$$\gamma^2 - 1 = (\gamma+1)(\gamma-1) \simeq 2(\gamma-1) \tag{26}$$

Again, the left-hand integral is trivial and reduces to

$$\frac{1}{2}\ln\left(\frac{\gamma-1}{\gamma_0-1}\right) \tag{27}$$

But, defining the kinetic energy $K = E - mc^2$

$$K = (\gamma-1)mc^2 \tag{28}$$

Hence,

$$\ln\left(\frac{K}{K_0}\right) = -\frac{4q^4B^2t}{3m^3c^5} \tag{29}$$

$$K(t) = K(t=0)\exp\left(-\frac{4q^4B^2t}{3m^3c^5}\right) \tag{30}$$

d) The acceleration in this case is such that $(d\gamma/dt = 0)$

$$\dot{\beta}^2 = \frac{q^2\beta^2B^2\sin^2\theta}{\gamma^2m^2c^2} \tag{31}$$

where θ is the angle between the velocity and the magnetic field. Using the general expressio for $P(t')$ for acceleration perpendicular to the velocity, we get

$$P(t') = \frac{2q^4P_T^2B^2}{3m^4c^5} \tag{32}$$

where $P_T = \gamma mv\sin\theta$ is the transverse momentum with respect to the magnetic field.

Now, the particle moves in a changing magnetic field, since the dipole field is not constant. However, assuming that the field changes slowly, we may use the adiabatic invariant described in section 12.5 of Jackson. Namely,

$$\frac{P_T^2}{B} \tag{33}$$

is an adiabatic invariant. Thus, we can write

$$P(t') = \frac{2q^4}{3m^4c^5}\left(\frac{P_T^2}{B}\right)\ B^3 \tag{34}$$

and conclude that during the trajectory of the particle, it radiates with a power $P(t')$ proportional to the third power of B.

Problem 4.

The generic expression of P is

$$P = \frac{2q^2}{3c^3}\gamma^6\left[\left(\vec{\beta}\ \dot{\vec{u}}\right)^2 + \gamma^{-2}\dot{\vec{u}}^2\right] \tag{35}$$

The general relation between the acceleration and the Force may be obtained by the relation:

$$\vec{F} = \frac{d(m\gamma\vec{u})}{dt} \tag{36}$$

and remembering that the variation of the energy of the particle,

$$\frac{dm\gamma c^2}{dt} = \vec{F}\vec{u} \tag{37}$$

Therefore, after some simple algera

$$\frac{d\vec{u}}{dt} = \frac{1}{m\gamma}\left[\vec{F} - \frac{\vec{u}(\vec{F}\vec{u})}{c^2}\right] \tag{38}$$

Therefore,

$$\dot{\vec{u}}^2 = \frac{1}{m^2\gamma^2}\left[\vec{F}^2 - \left(1 + \frac{1}{\gamma^2}\right)(\vec{F}\vec{u})^2\right] \tag{39}$$

while

$$\left(\vec{\beta}\dot{\vec{u}}\right)^2 = \frac{1}{m^2\gamma^6}(\vec{F}\vec{u})^2 \tag{40}$$

Finally, putting everything together we get

$$P = \frac{2q^2\gamma^2}{3m^2c^3}\left[\vec{F}^2 - (\vec{F}\vec{u})^2\right] \tag{41}$$

For the case at hand, the force is just the lorentz force given by

$$\vec{F} = q(\vec{E} + \vec{u} \text{ x } \vec{B}) \tag{42}$$

and then, the result of the excercise is obvious.

b) To express the energy radiated in terms of a Lorentz invariant, one just has to remember that one can relate the Lorentz Force to the four vector

$$\frac{dP^\mu}{d\tau} = qF^{\mu\nu}U_\nu = \gamma\left(q\vec{E}\vec{u}, q(\vec{E} + \vec{\beta} \text{ x } \vec{B})\right) \tag{43}$$

where

$$P^\mu = mcU^\mu \tag{44}$$

The result of the excercise is then obvious, and just reduces to a Lorentz contraction.