A COMPACT FORMULA FOR ROTATIONS AS SPIN MATRIX POLYNOMIALS

▶ Why bother? Consider the rotation matrix for 2-spinors,

$$\boxed{e^{ia(\hat{n}\cdot\vec{\sigma})} = I\cos a + i(\hat{n}\cdot\vec{\sigma})\sin a}\ ,$$

so the generic SU(2) (SO(3)) group element multiplication is

$$e^{ia(\hat{n}\cdot\vec{\sigma})}e^{ib(\hat{m}\cdot\vec{\sigma})}$$

$$= I(\cos a\cos b - \hat{n}\cdot\hat{m}\sin a\sin b) + i(\hat{n}\sin a\cos b + \hat{m}\sin b\cos a - \hat{n}\times\hat{m}\ \sin a\sin b)\cdot\vec{\sigma}$$

$$= I\cos c + i(\hat{k}\cdot\vec{\sigma})\sin c = e^{ic(\hat{k}\cdot\vec{\sigma})}.$$

Manifestly, $\cos c = \cos a\cos b - \hat{n}\cdot\hat{m}\sin a\sin b$, the spherical law of cosines. Given c, $\rightsquigarrow$

$$\boxed{e^{ic\hat{k}\cdot\vec{\sigma}} = \exp\left(i\tfrac{c}{\sin c}(\hat{n}\sin a\cos b + \hat{m}\sin b\cos a - \hat{n}\times\hat{m}\ \sin a\sin b)\cdot\vec{\sigma}\right)}$$

✓**Composition law of finite rotations** —J W Gibbs, 1884.

▶ For **general representations** of SU(2), ($[J_a, J_b] = i\epsilon_{abc}J_c$), the rotation matrices $e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})}$ are $(2j+1)\times(2j+1)$ dimensional for either integral spin, $j = 0,\ 1,\ 2,\ 3,\ \cdots$, or half-integral spin, $j = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \cdots$, and depend on the rotation angle θ around the unit vector axis $\hat{\boldsymbol{n}}$.

The third component of the matrix triplet $\boldsymbol{J}$ in standard convention is the diagonal matrix $J_3 = \mathrm{diag}(j, j-1, j-2, \cdots, 1-j, -j)$.

By the Cayley–Hamilton theorem, the expansion of this exponential in powers of $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$ may be recast as a **finite sum of powers** of $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$, the highest power being of order $2j$. (These polynomials of Lie generators are in the universal enveloping algebra of $su(2)$.)
We just saw $j = 1/2$, so $\boldsymbol{J} = \boldsymbol{\sigma}/2$,

$$e^{i(\theta/2)(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma})} = I_2 \cos\theta/2 + i(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma})\sin\theta/2 \ ,$$

and we also know the (vector) Rodrigues' rotation formula (Euler 1770) for the triplet, $j = 1$, so $J_3 = \mathrm{diag}(1, 0, -1)$,

$$\begin{aligned} e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} &= I_3 + i(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})\sin\theta + (\hat{\boldsymbol{n}}\cdot\boldsymbol{J})^2(\cos\theta - 1) \\ &= I_3 + (2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))\cos(\theta/2) + \tfrac{1}{2}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^2 \ . \end{aligned}$$

but ... ↬ **what about higher representations**?

People have computed low-lying representations since the mid 30s (Wigner 1931, Lehrer-Ilamed 1964,van Wageningen 1964, Weinberg 1964, Weber & Williams 1965, Zemach 1965, Torruella 1975) — but have failed to find a closed expression for arbitrary representations.

We found one:

$$\boxed{e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} = \sum_{k=0}^{2j} \frac{c_k(\theta)}{k!}\left(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin\frac{\theta}{2}\right)^k} \ ,$$

where, with use of the floor function,

$$c_k(\theta) = (\cos(\theta/2))^\epsilon \ \ \underset{\lfloor j-k/2\rfloor}{\text{Trunc}}\left(\frac{(\arcsin\sqrt{x}/\sqrt{x})^k}{(\sqrt{1-x})^\epsilon}\right),$$

with $x = \sin^2(\theta/2)$ and

$$\epsilon(j-k/2) = 2j-k-2\lfloor j-k/2\rfloor = \frac{1-(-1)^{2j-k}}{2} \ ,$$

a binary **parity variable**: 0 for even $2j-k$, and 1 for odd $2j-k$.

⊛ The effective variable $2j-k$ is the **descending order** of this polynomial.

↬ Trunc_n is a Taylor polynomial in x, i.e., it denotes truncating the infinite series of its arguments to $\mathrm{O}(x^n)$,

$$\underset{n}{\text{Trunc}}\left(\sum_{m=0}^{\infty} a_m x^m\right) \equiv \sum_{m=0}^{n} a_m x^m \ .$$

For the quartet, $j = 3/2$,

$$
\begin{aligned}
e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} &= I_4 \cos(\theta/2)\left(1 + \tfrac{1}{2}\sin^2(\theta/2)\right) \\
&+ (2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))\left(1 + \tfrac{1}{6}\sin^2(\theta/2)\right) \\
&+ \frac{1}{2!}\Big(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2)\Big)^2\cos(\theta/2) \;+ \frac{1}{3!}\Big(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2)\Big)^3 \;.
\end{aligned}
$$

For the quintet, $j = 2$,

$$
\begin{aligned}
e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} &= I_5 + (2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))\cos(\theta/2)\left(1 + \tfrac{2}{3}\sin^2(\theta/2)\right) \\
&+ \frac{1}{2!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^2\left(1 + \tfrac{1}{3}\sin^2(\theta/2)\right) \\
&+ \frac{1}{3!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^3\cos(\theta/2) + \frac{1}{4!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^4 \;.
\end{aligned}
$$

For the sextet, $j = 5/2$,

$$
\begin{aligned}
e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} &= I_6 \cos(\theta/2)\left(1 + \tfrac{1}{2}\sin^2(\theta/2) + \tfrac{3}{8}\sin^4(\theta/2)\right) \\
&+ (2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))\left(1 + \tfrac{1}{6}\sin^2(\theta/2) + \tfrac{3}{40}\sin^4(\theta/2)\right) \\
&+ \frac{1}{2!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^2\cos(\theta/2)\left(1 + \tfrac{5}{6}\sin^2(\theta/2)\right) \\
&+ \frac{1}{3!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^3\left(1 + \tfrac{1}{2}\sin^2(\theta/2)\right) \\
&+ \frac{1}{4!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^4\cos(\theta/2) + \frac{1}{5!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^5 \ .
\end{aligned}
$$

For spin $j = 5$,

$$
\begin{aligned}
&e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} = I_{11} \\
&+ (2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))\cos(\theta/2)\left(1 + \tfrac{2}{3}\sin^2(\theta/2) + \tfrac{8}{15}\sin^4(\theta/2) + \tfrac{16}{35}\sin^6(\theta/2) + \tfrac{128}{315}\sin^8(\theta/2)\right) \\
&+ \tfrac{1}{2!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^2\left(1 + \tfrac{1}{3}\sin^2(\theta/2) + \tfrac{8}{45}\sin^4(\theta/2) + \tfrac{4}{35}\sin^6(\theta/2) + \tfrac{128}{1575}\sin^8(\theta/2)\right) \\
&+ \tfrac{1}{3!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^3\cos(\theta/2)\left(1 + \sin^2(\theta/2) + \tfrac{14}{15}\sin^4(\theta/2) + \tfrac{164}{189}\sin^6(\theta/2)\right) \\
&+ \tfrac{1}{4!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^4\left(1 + \tfrac{2}{3}\sin^2(\theta/2) + \tfrac{7}{15}\sin^4(\theta/2) + \tfrac{328}{945}\sin^6(\theta/2)\right) \\
&+ \tfrac{1}{5!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^5\cos(\theta/2)\left(1 + \tfrac{4}{3}\sin^2(\theta/2) + \tfrac{13}{9}\sin^4(\theta/2)\right) \\
&+ \tfrac{1}{6!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^6\left(1 + \sin^2(\theta/2) + \tfrac{13}{15}\sin^4(\theta/2)\right) \\
&+ \tfrac{1}{7!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^7\cos(\theta/2)\left(1 + \tfrac{5}{3}\sin^2(\theta/2)\right) \\
&+ \tfrac{1}{8!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^8\left(1 + \tfrac{4}{3}\sin^2(\theta/2)\right) \\
&+ \tfrac{1}{9!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^9\cos(\theta/2) \\
&+ \tfrac{1}{10!}(2i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\sin(\theta/2))^{10} \ ,
\end{aligned}
$$

etc.

⊛ The trace of each exponential $(2j+1)\times(2j+1)$ matrix is the **character** of this spin j representation of $SU(2)$: the Gegenbauer and also the 2nd kind Chebyshev polynomial,
$C^1_{2j}(\cos(\theta/2)) = U_{2j}(\cos(\theta/2)) = \sin((2j+1)\theta/2)/\sin(\theta/2)$.

$\rightsquigarrow$ Each term in the Taylor polynomials $\mathrm{Trunc}_n\ldots$ is positive semi-definite—no oscillations!

Taking k derivatives of this expansion with respect to θ and evaluating at $\theta = 0 \Longrightarrow$ the $(i\hat{\boldsymbol{n}}\cdot\boldsymbol{J})^k$ term in the series is selected with unit coefficient— and all other terms must vanish. It is also necessary that the last term be of order $2j$.

▶ All terms, for integral j are actually periodic functions θ, i.e. they can be recast in trigonometric functions of θ, and are thus periodic in 2π. By contrast, for half-integral spin j, all coefficients cannot (they are only trigonometric functions of $\theta/2$), and and thus have period 4π, instead. Fermions and bosons flip relative signs at π.

¶ For a given j, the matrix coefficients of the terms of $\sin^{2j}(\theta/2)$ and $\cos(\theta/2)\sin^{2j-1}(\theta/2)$ in the expansion, polynomials of order $(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})^{2j}$ and $(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})^{2j-1}$ respectively, must vanish **for all eigenvalues** of J_3 **except the extremal ones**, $\pm j$. That is, they are proportional to the characteristic polynomial of $J_3[2j-1]$ and hence of $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}[2j-1]$. Constraining these coefficients to zero, which amounts to projecting out these extremal eigenvalues, necessarily reduces the expansion of $e^{i\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J}[j]}$ to that of $e^{i\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J}[j-1]}$.

⊙ The alternating binary parities ϵ, even (0), or odd (1—which inserts a factor $\cos(\theta/2)$ in the coefficients and effectively in the denominator of the truncated series), **interleave** for a given j; and their location shifts by one for a given k, going from integral to half-integral spins.

⇝ When the large j limit is considered, the resulting expression would be expected to resemble the expansion of a scalar exponential, as the Cayley–Hamilton theorem applied to higher-order terms provides dwindling corrections—provided the requisite periodicities be respected!

↬In general, the coefficients (including the $\sin\theta/2$s) would then be **periodicized monomials** $(\theta/2)^k$ with the suitable periodicity.

▼ The leading coefficient (to the identity) of the expansion, c_0, is always just 1, for even ϵ, so then for all integral spins, large and small. In striking contrast, for odd ϵ, large half-integral spins j, the leading term tends to

$$I_{2j+1}\ \mathrm{sgn}\Big(\cos(\theta/2)\Big)\ ,$$

a square waveform with the required periodicity of 4π. It agrees with the integral spin in $[-\pi,\pi]$, but flips sign outside this interval.

♦ Similarly, the second term in the expansion linear in the Lie algebra generators, for large integral spins has odd bimodal parity ϵ and tends to

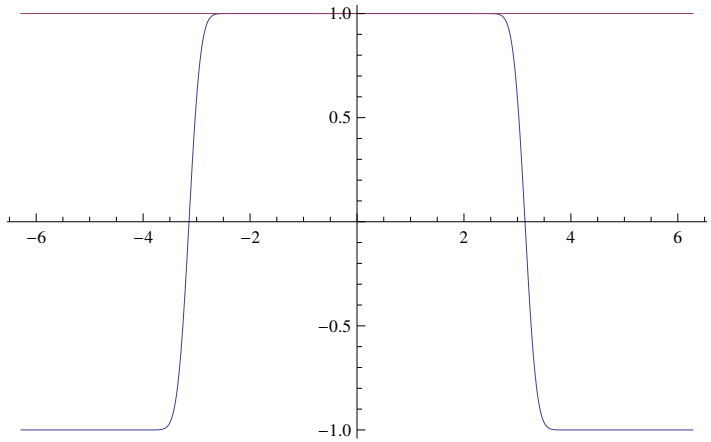
$$i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\Big(\theta-2\pi\lfloor\tfrac{\theta}{2\pi}-\tfrac{1}{2}\rfloor\Big)\ ,$$

a sawtooth forced to maintain periodicity in 2π.

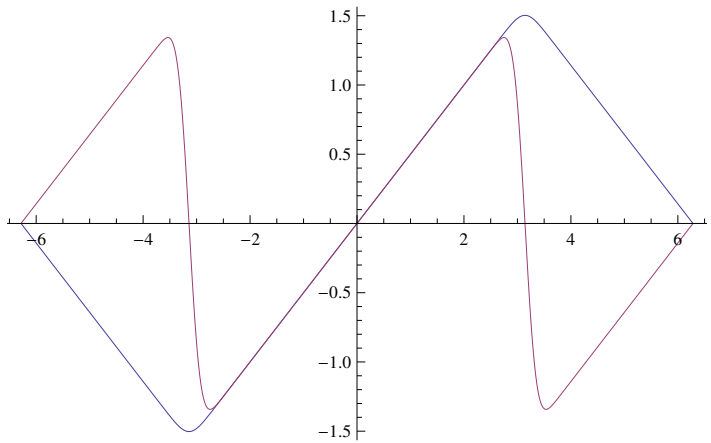
⟲ By contrast, for large half-integral spins, even bimodal parity, the limiting triangular waveform can be more symmetric, as the slope of the linear function must reverse at the boundary of $[-\pi,\pi]$,

$$i\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\ \mathrm{sgn}\Big(\cos(\theta/2)\Big)\ \Big(\theta-2\pi\lfloor\tfrac{\theta}{2\pi}-\tfrac{1}{2}\rfloor\Big)\ .$$

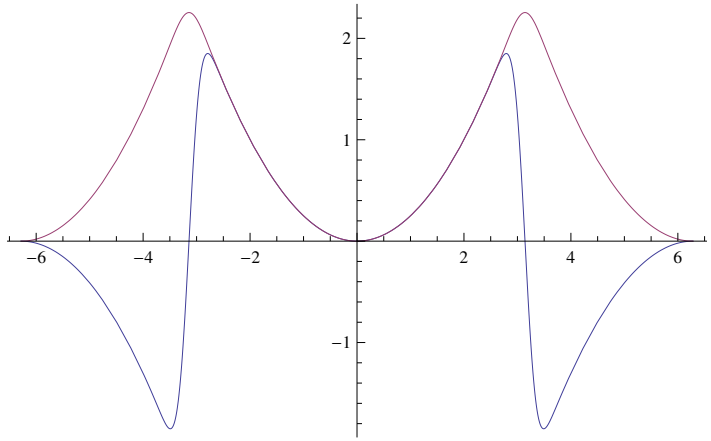
$-2\pi \leq \theta \leq 2\pi$, for spins $j = 137/2$ and $j = 69$.



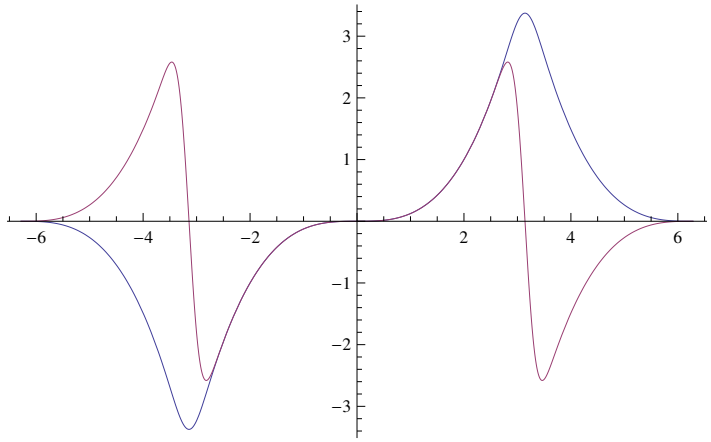
c_0 versus θ



$c_1 \sin(\theta/2)$ versus θ



$c_2 \sin^2(\theta/2)$ versus θ



$c_3 \sin^3(\theta/2)$ versus θ

⋔ There is a plethora of **derivations**, by now. They normally rely on the **Lagrange–Sylvester** polynomial expansion of analytic matrix functions with **Vandermonde matrix inversion** and/or the Cayley–Hamilton theorem re-expressed through central factorial polynomials.

Specifically,

$$f\left(\mathbb{M}\right) = \sum_{i=1}^{N} f\left(\lambda_i\right) \quad \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{\mathbb{M}-\lambda_j}{\lambda_i-\lambda_j} = \sum_{m=0}^{N-1} C_m\left[f\right] \ \mathbb{M}^m \ ,$$

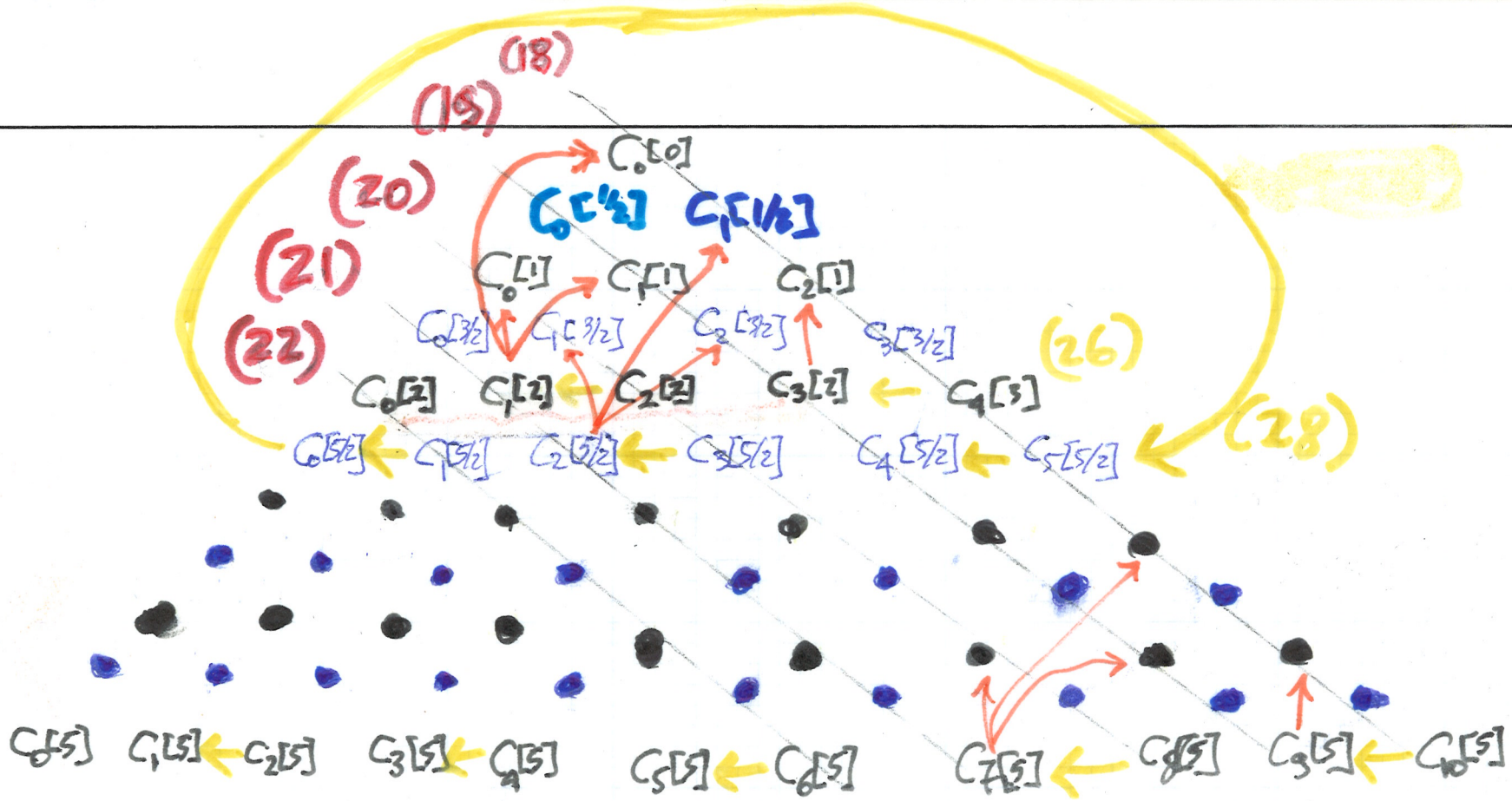
$$f\left(\lambda_k\right) = \sum_{m=0}^{N-1} C_m\left[f\right] \ \left(\lambda_k\right)^m \equiv V\left[\mathbb{M}\right]_{km} \ C_m\left[f\right] \ ,$$

where the $N \times N$ Vandermonde matrix of $\mathbb{M}$ **nondegenerate** eigenvalues ($\rightsquigarrow$ invertible) is

$$V\left[\mathbb{M}\right] = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{pmatrix} \ .$$

Hence

$$C_m\left[f\right] = V\left[\mathbb{M}\right]^{-1}_{mk} \ f\left(\lambda_k\right) \ ,$$



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The Curtright Pyramid

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In practice, we evaluated

$$
\begin{aligned}
e^{i\theta(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} = {} & e^{i\theta j}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j+1}{1}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j+2}{2}\ \cdots \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}+j}{2j} \\
& + e^{i\theta(j-1)}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j}{-1}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j+2}{1}\ \cdots \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}+j}{2j-1} \\
& + \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
& + e^{i\theta(1-j)}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j}{1-2j}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j+1}{2-2j}\ \cdots \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}+j}{1} \\
& + e^{-i\theta j}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j}{-2j}\ \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}-j+1}{1-2j}\ \cdots \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{J}+j-1}{-1}\ .
\end{aligned}
$$

Also see T Curtright & T van Kortryk, *JPhys* **A48** (2015) 025202