TERNARY PLOTS FOR NEUTRINO MIXING

▶ The unitary PMNS neutrino matrix U mixes left-handed fields of the three neutrino mass eigenstates ν_j, $i = 1, 2, 3$, into lepton-flavor linear combinations ν_l, $l = e, \mu, \tau$, named after the charged leptons they couple to,

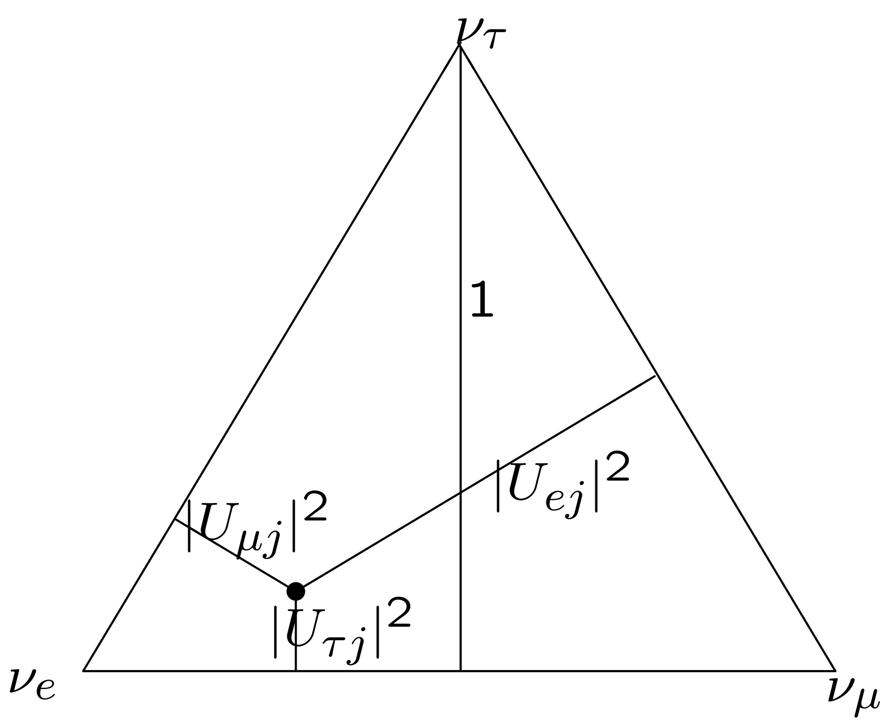
$$\boxed{\nu_l = \sum_{j=1}^{3} U_{lj}\ \nu_j} \ .$$

▶All three angles in quark mixing are **small**: $\theta_{12} \approx \pi/14 \approx \lambda \approx 0.23$, $\theta_{23} \approx \lambda^2 \approx 0.04$, $\theta_{13} \approx \lambda^3 \approx 0.01$. By sharp contrast, for neutrino mixing, the reactor one is small, $\theta_{13} \approx 0.16 \pm 0.02$, but **two are big**, the solar $\theta_{12} \approx (\pi/4 - 0.19) \pm 0.02$, and the atmospheric $\theta_{23} \approx (\pi/4 + 0.00) \pm 0.09$. "Popsicle plots", where the relative three flavor contents are depicted by different color sections adding up to a constant, do not make evident how they contrast among themselves.

✓However, the **unitarity constraints** of PMNS composition vectors

$$N_j = (|U_{ej}|^2, |U_{\mu j}|^2, |U_{\tau j}|^2)$$

are incorporated automatically in **ternary plots** (Dalitz plots): barycentric coordinates allow three variables with a fixed sum to be plotted as mere **points** inside an equilateral triangle on a plane and to be thus visually compared collectively. Consider the plane equilateral triangle with sides $a = 2/\sqrt{3}$ and thus height 1.



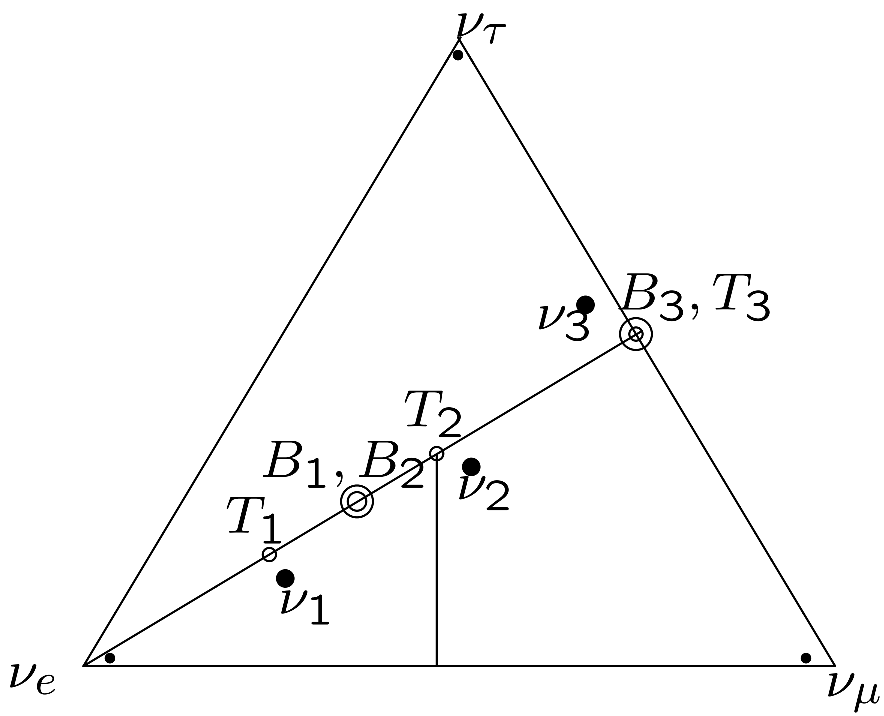
↬ Every point inside this equilateral triangle has its perpendiculars to the three sides sum to 1, often referred to as "Viviani's theorem":

this is self-evident, as the point divides the equilateral triangle into three triangles with the same base, a (the sides of the equilateral), and the perpendiculars as their heights.

$\circledast$Since the areas of the three triangles make up the area of the equilateral, the three heights sum up to the height of the equilateral, for any point in the triangle.

$\rightsquigarrow$ Since the angles of the equilateral are all $\pi/3$, the Cartesian coordinates of the point representing the vector N_j are $(x, y) = \left(\frac{2|U_{\mu j}|^2+|U_{\tau j}|^2}{\sqrt{3}}, |U_{\tau j}|^2\right)$.

The origin $(x, y) = (0, 0)$ then amounts to ν_e.



▶ The components of the constrained 3-vectors $N_j = (|U_{ej}|^2, |U_{\mu j}|^2, |U_{\tau j}|^2)$, (large solid circles),

$$N_1 = (0.67, 0.20, 0.14), N_2 = (0.31, 0.40, 0.30), N_3 = (0.03, 0.41, 0.57) \; .$$

↬The (unrealistic) B_j (bimaximal), T_j (tribimaximal) benchmark mixing paradigms require $\theta_{13} = 0$ and maximal $\theta_{23} = \pi/4$; thus, they lie on the $\nu_\mu \leftrightarrow \nu_\tau$ approximate symmetry axis. The midpoint T_2 is the center of the equilateral triangle. Large empty circles,

$$B_1 = (1/2, 1/4, 1/4), \quad B_2 = (1/2, 1/4, 1/4), \quad B_3 = (\; 0 \; , \; 1/2, 1/2) \; ,$$

and small empty circles,

$$T_1 = (2/3, 1/6, 1/6), \quad T_2 = (1/3, 1/3, 1/3), \quad T_3 = (\ 0\ ,\ 1/2, 1/2)\ .$$

The small solid circles near the vertices represent the three quarks. Since the largest angle, the Cabbibo angle, is of the order of magnitude of θ_{13}, they are clustered near the vertices of the triangle by amounts comparable to the offset of ν_3 from the triangle side.

$\looparrowright$Such collective contrasts would be **unwieldy in tabular or popsicle plots**.

$\rightsquigarrow$ Ability to visualize mixing models.

$\circlearrowleft$ Assuming unitarity, variation of the hypothetical CP-violation phases δ_{CP} will generate **small motions** on short curves of the three points N_j inside the triangle in unison: recall that the unitarity constrains the sum of the verticals of all three points N_j on each side of the triangle **separately to also equal one**,
$|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2 = 1$;
and likewise for the second, μ, and third, τ.

▶ The plot may thus be of utility in surveying the collective sensitivity of such determinations on CP violation. Perhaps more significantly, the important task of comparing with scores of more realistic theoretical texture schemes could be addressed more efficiently and compactly in a graphical manner, rather than a tabular form.