BRANES AND QUANTUM NAMBU BRACKETS

Consider an open topological membrane 3-form action,

$$S = \int (z^1\, dz^2 \wedge dz^3 \wedge dz^4 + L_1\, dL_2 \wedge dL_3 \wedge dt).$$

• Originates in an exact 4-form (analogous to the Hamiltonian symplectic 2-form), $dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 + dL_1 \wedge dL_2 \wedge dL_3 \wedge dt$, evaluated on the boundary of a 4-manifold, like a 3-dim WZW interaction term. In coordinates,

$$S = \int dt d\alpha d\beta \, \left(\frac{\epsilon^{ijkl}}{4} z^i \partial_t z^j \partial_\alpha z^k \partial_\beta z^l + L_1 (\partial_\alpha L_2 \partial_\beta L_3 - [\beta\alpha]) \right) .$$

The variational eqns of motion resulting from δz^i are

$$\frac{dz^l}{dt} = \epsilon^{lijk} \partial_i L_1 \partial_j L_2 \partial_k L_3 = \frac{\partial(z^l, L_1, L_2, L_3)}{\partial(z^1, z^2, z^3, z^4)} ,$$

a Jacobian determinant (volume element).

$\rightsquigarrow$ Instead of Hamilton's eqns, the classical eqns of motion are

$$\boxed{\dot{z}^l = \{ z^l, L_1, L_2, L_3 \}},$$

the celebrated **Nambu Bracket** (1973).

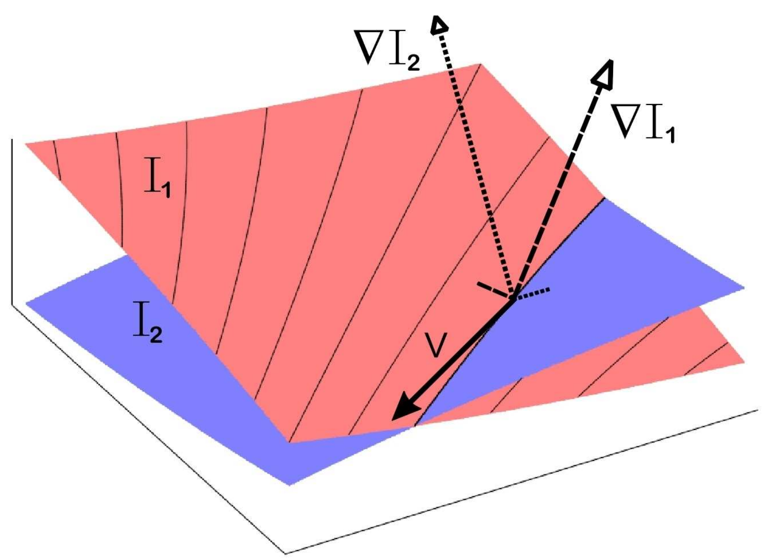
• It generalizes and supplants the Poisson Bracket, and is likewise linear and antisymmetric in its arguments. (Here, 3 "Hamiltonians", instead of one.)

In general, the classical motion of Maximally Superintegrable Systems in phase space, $z^i = (x, p_x, y, p_y, ...)$, cannot **avoid** being described by NBs.

For N degrees of freedom, $\rightsquigarrow$ $2N$-dimensional phase space, if there are extra invariants beyond the N required for integrability, the system is called superintegrable.

• At most, there are $2N - 1$ algebraically independent integrals of motion: Maximal Superintegrability.

Motion is confined in phase space on the constant surfaces specified by these integrals $\rightsquigarrow$ the phase-space velocity $\mathbf{v} = (\dot{\mathbf{q}}, \dot{\mathbf{p}})$ is always perpendicular to the $2N$-dim **phase space gradients** $\nabla = (\partial_{\mathrm{q}}, \partial_{\mathrm{p}})$ of all these integrals of the motion:



$\Longrightarrow$ The phase-space velocity must be proportional to the cross-product of **all** those gradients.

$\rightsquigarrow$ For any phase-space function $k(\mathbf{q},\mathbf{p})$, the motion is fully specified by the (NB) phase-space Jacobian,

$$\begin{aligned}
\frac{dk}{dt} &= \nabla k \cdot \mathbf{v} \\
&\propto \partial_{i_1} k\, \epsilon^{i_1 i_2 \dots i_{2N}}\, \partial_{i_2} L_1 \dots \partial_{i_{2N}} L_{2N-1} \\
&= \frac{\partial(k, L_1, \dots, \dots, L_{2N-1})}{\partial(q_1, p_1, q_2, p_2, \dots, q_N, p_N)} \\
&\equiv \{k, L_1, \dots, L_{2N-1}\}\ .
\end{aligned}$$

- The proportionality constant is shown to be time-invariant.

$\rightsquigarrow$ The flow is divergenceless, $\nabla \cdot \mathbf{v} = 0$ (Liouville's thm).

eg, S^2: $H = \frac{1}{2}(L_x L_x + L_y L_y + L_z L_z)$; $L_z = x p_y - y p_x$, $L_y = -\sqrt{1-x^2-y^2}\ p_x$, $L_x = \sqrt{1-x^2-y^2}\ p_y$; $\{L_x, L_y\} = L_z$, etc,

$$\boxed{\frac{dk}{dt} = \frac{\partial\big(k, L_x, L_y, L_z\big)}{\partial\big(x, p_x, y, p_y\big)}}\ .$$

eg, S^N: $H = \frac{1}{2}\sum_{a=1}^{N} P_a P_a + \frac{1}{4}\sum_{a,b=1}^{N} L_{ab} L_{ab}$,
$P_a = \sqrt{1-q^2}\, p_a$, for $a = 1, \cdots, N$, and $L_{a,b} = q^a p_b - q^b p_a$,

$$\frac{dk}{dt} = \frac{(-1)^{(N-1)}}{P_2 P_3 \cdots P_{N-1}}\ \frac{\partial\left(k, P_1, L_{12}, P_2, L_{23}, P_3, \cdots, P_{N-1}, L_{N-1\ N}, P_N\right)}{\partial\left(x_1, p_1, x_2, p_2, \cdots, x_N, p_N\right)}\ .$$

eg, oscillators, chiral models, Coulomb (Hydrogen atom), etc

• In general, NBs possess all antisymmetries of Jacobian determinants; and obey the Leibniz rule,

$$\{k(A,B),L_1,L_2,...\} = \frac{\partial k}{\partial A}\{A,L_1,L_2,...\} + \frac{\partial k}{\partial B}\{B,L_1,L_2,...\}.$$

$\rightsquigarrow$ Eg, hamiltonians are time-invariant,

$$\frac{dH}{dt} = \left\{\frac{\mathbf{L}\cdot\mathbf{L}}{2}, L_x, L_y, L_z\right\} = 0.$$

• Maximal even-rank classical NBs resolve into products of Poisson Brackets. Eg,

$$\{A,B,C,D\} = \{A,B\}\{C,D\} - \{A,C\}\{B,D\} - \{A,D\}\{C,B\},$$

in comportance with full antisymmetry under permutations of A, B, C, and D.

$\Longrightarrow$ the specific S^2 membrane action yields the same eqns of motion as a particle action, $S = \int dt\, \left(\dot{x}p_x + \dot{y}p_y - \frac{L\cdot L}{2}\right)$!

• This resolution is a general result: all even-NBs are Pfaffians of the (antisymmetric) matrix with elements $\{A_i, A_j\}$:

$$\{A_i, A_j, ..., A_k, A_l\} \ \propto\ \epsilon^{i,j,...,k,l}\{A_i,A_j\}...\{A_k,A_l\}.$$

• The impossibility to antisymmetrize more than $2N$ indices in $2N$-dimensional phase space,

$$\epsilon^{ab....c[i}\epsilon^{j_1 j_2 \cdots j_{2N}]} = 0\ ,$$

leads to the (generalized) “Fundamental” Identity (FI),

$$\{V\{A_1,...,A_{m-1},A_m\},A_{m+1},...,A_{2m-1}\} + \{A_m,V\{A_1,...,A_{m-1},A_{m+1}\},A_{m+2},...,A_{2m-1}\}$$
$$+... + \{A_m,...,A_{2m-2},V\{A_1,...,A_{m-1},A_{2m-1}\}\} = \{A_1,...,A_{m-1},V\{A_m,A_{m+1},...,A_{2m-1}\}\}.$$

• $(2m-1)$-elements, (+1) V, $(m+1)$-terms.

• **Not the generalization of the Jacobi Identity** as an encoding of associativity—only a consequence of the derivation property of NBs, instead, $\delta A = \{A, B, \cdots\}$. If Leibniz's rule holds,

$$\delta(A\mathcal{A}) = A\delta\mathcal{A} + \mathcal{A}\delta A = A\{\mathcal{A}, B, \cdots\} + \mathcal{A}\{A, B, \cdots\},$$

$\Longrightarrow$

$$\delta\{C, D, \cdots\} = \{\delta C, D, \cdots\} + \{C, \delta D, \cdots\} + \cdots \ ,$$

$\Longrightarrow$ FI

$$\{\{C, D, \cdots\}, B, \cdots\} = \{\{C, B, \cdots\}, D, \cdots\} + \{C, \{D, B, \cdots\}, \cdots\} + \cdots \ .$$

(But this need not hold upon quantization.)

The proportionality constant V

$$\frac{dA}{dt} = V\{A, L_1, \cdots, L_{2N-1}\},$$

must be a time-invariant if it has no **explicit** time dependence, since

$$\frac{d}{dt}\Big(V\{A_1, \cdots, A_{2N}\}\Big)$$

$$= \dot{V}\{A_1, \cdots, A_{2N}\} + V\{\dot{A}_1, \cdots, A_{2N}\} + \cdots + V\{A_1, \cdots, \dot{A}_{2N}\}.$$

$\Longrightarrow$

$$V\{V\{A_1, \cdots, A_{2N}\}, L_1, \cdots, L_{2N-1}\} = \dot{V}\{A_1, \cdots, A_{2N}\}$$

$$+V\{V\{A_1, L_1, \cdots, L_{2N-1}\}, \cdots, A_{2N}\} + \cdots + V\{A_1, \cdots, V\{A_{2N}, L_1, \cdots, L_{2N-1}\}\}.$$

$$\Longrightarrow \qquad \frac{dV}{dt} = 0 \, .$$

QUANTIZATION

Deform classical structures to operator ones. Undeserved bad reputation, on account of top-down shortcomings. There are consistency complications, but not debilitating ones.

• A useful check: NB quantization **must** coincide with standard hamiltonian quantization for specific models. [Zachos & Curtright, New J Phys **4** (2002) 83.1-83.16]

Nambu's (1973) proposal QNBs:

$$[A,B] \equiv AB - BA,$$

$$[A,B,C] \equiv ABC - ACB + BCA - BAC + CAB - CBA,$$

$$[A,B,C,D] \equiv A[B,C,D] - B\,[C,D,A] + C\,[D,A,B] - D\,[A,B,C] =$$
$$= [A,B][C,D] + [A,C][D,B] + [A,D][B,C]$$
$$+[C,D][A,B] + [D,B][A,C] + [B,C][A,D]\ ,$$

etc.

Even QNBs resolve into strings of commutators, "Quantum Pfaffians", $\boxed{\propto \epsilon^{ij..kl}[A_i,A_j]...[A_k,A_l]}$.

Good classical limit: $[A_1,...,A_{2n}] \ \rightarrow\ n!(i\hbar)^n\{A_1,...,A_{2n}\}$ as $\hbar \rightarrow 0$. By contrast, odd QNBs have a bad classical limit.

Full antisymmetry, but no Leibniz property or FI, in general. Only a subjective shortcoming, dependent on the specific application context! Quantization is consistent.

• QNBs do satisfy the celebrated fully antisymmetric **Generalized Jacobi Identity** (Hanlon & Wachs; Azcárraga, Izquierdo, Perelomov & Pérez Bueno), which encodes **associativity**. Eg, for 4-QNBs,

$$[[A,B,C,D],E,F,G] + \quad permutations \quad = 0,$$

a total of $\frac{7!}{3!4!} = 35$ terms, instead of the FI's 5.

$$\epsilon^{ijklmnr}[[A_i,A_j,A_k,A_l],A_m,A_n,A_r] = 0.$$

Eg, objectively, for S^2,

$$\boxed{\frac{dk}{dt} = \frac{1}{i\hbar}[k,H] = \frac{-1}{2\hbar^2}\,[k,L_x,L_y,L_z]}\;,$$

a derivation (an exceptional situation). $\rightsquigarrow$ **Here**, in phase space, even Leibniz and FI hold, **nevertheless**. Good $\hbar \to 0$ limit.

NB. For $A \propto \mathbb{1}$, thus $dA/dt = 0$, $[\mathbb{1},B,C,D] = 0$ holds identically, in contrast to the 3-argument QNB, $[\mathbb{1},B,C] = [B,C] \neq 0$. Thus, **no debilitating constraint** among the arguments B,C,D is imposed; the inconsistency identified originally is a feature of odd-argument QNBs (which fail as deformations of CNBs: they lack a good classical limit); but does not restrict the even-argument QNBs of phase space. Instead, odd CNBs are reachable from larger, even QNBs like the ones discussed: Eg, $[A,B,C,p_y] \to -2\hbar^2\{A,B,C\}$.

• More generic situation, eg for S^N, $N>2$:
the QNBs provide the correct quantization rule, but **need not** satisfy the naive Leibniz property (and FI) for consistency, as they are not necessarily plain derivations; instead, **time derivatives are entwined** inside strings of invariants.
Eg, for S^3,

$$\Big[k,P_1,L_{12},P_2,L_{23},P_3\Big] = 3\hbar^2\Big(P_2[k,H]+[k,H]P_2\Big) + \mathcal{Q}(O(\hbar^5)).$$

$\leadsto$

$$\Big[k, P_1, L_{12}, P_2, L_{23}, P_3\Big] = 3i\hbar^3 \frac{d}{dt}\Big(P_2 k + k P_2\Big) + \mathcal{Q}(O(\hbar^5)).$$

The right hand side is **not an unadorned derivation** on k
$\leadsto$ does not impose a Leibniz rule on the left hand side. (Other consistency constraints are more suitable and are, of course, satisfied.)
$\mathcal{Q}(O(\hbar^5))$ is a nested commutator "quantum rotation".

• A lark: Ignoring $\mathcal{Q}$, could one eschew solving the Jordan-Kurosh spectral problem,

$$\Big[k, P_1, L_{12}, P_2, L_{23}, P_3\Big] \sim 3i\hbar^3 \Big(P_2 \frac{dk}{dt} + \frac{dk}{dt} P_2\Big) \quad ?$$

formally, $\leadsto$

$$3i\hbar^3 \frac{dk}{dt} \sim \sum_{n=0}^{\infty} (-P_2)^n \Big[k, P_1, L_{12}, P_2, L_{23}, P_3\Big] (P_2)^{-n-1},$$

and so envision a **different** bracket which **is** a derivation?

• Kepler problem (Hydrogen atom): S^3, through Pauli-Runge-Lenz vector, classically, $\mathbf{A} = \mathbf{p}\times\mathbf{L} - \hat{\mathbf{r}}$, $\leadsto$ $\mathbf{A}\cdot\mathbf{L} = 0$; $\leadsto$ $H = \frac{\mathbf{A}^2 - 1}{2\mathbf{L}^2}$. Defining $\mathbf{D} \equiv \frac{\mathbf{A}}{\sqrt{-2H}}$, $\leadsto$ $\mathcal{R} = \mathbf{L} + \mathbf{D}$, $\mathcal{L} = \mathbf{L} - \mathbf{D}$, $\leadsto$ $SO(3)\times SO(3) \sim SO(4)$, $H = \frac{-1}{2\mathcal{R}^2} = \frac{-1}{2\mathcal{L}^2}$,

$$\frac{dk}{dt} = -H^2 \{k, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\}.$$

Quantized as above, $[\mathcal{R}_i, \mathcal{R}_j] = 2i\hbar\, \epsilon^{ijk}\mathcal{R}_k$, ...
$3i\hbar^3\Big((\mathcal{R}_3 + \mathcal{L}_3)\frac{dk}{dt} + \frac{dk}{dt}(\mathcal{R}_3 + \mathcal{L}_3)\Big) = H\ [k, \mathcal{R}_3 + \mathcal{L}_3, \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2]\ H + ...,$
with $H = \frac{-1}{2(\mathcal{R}^2 + \hbar^2)} \mapsto \frac{-1}{2\hbar^2(4s(s+1)+1)} = \frac{-1}{2\hbar^2(2s+1)^2}$, $s = 0, \frac{1}{2}, 1, ...$

WHAT HAVE WE LEARNED?

• General methodology, successful in a large number of systems:
As suggested by the classical NB, the commutator resolution of a suitably chosen QNB parallels the classical combinatorics to yield a commutator with the hamiltonian ($\rightsquigarrow$ time derivative), entwined with invariants.

In quantization, associativity trumps naive derivation features.

• • Quantum Nambu Brackets are consistent and describe the quantum behavior of superintegrable systems equivalently to standard hamiltonian quantization. Reputed inconsistencies have been addressing unsuitable (and untenable) conditions.

• Guide for more general systems—even non-Hamiltonian ones.