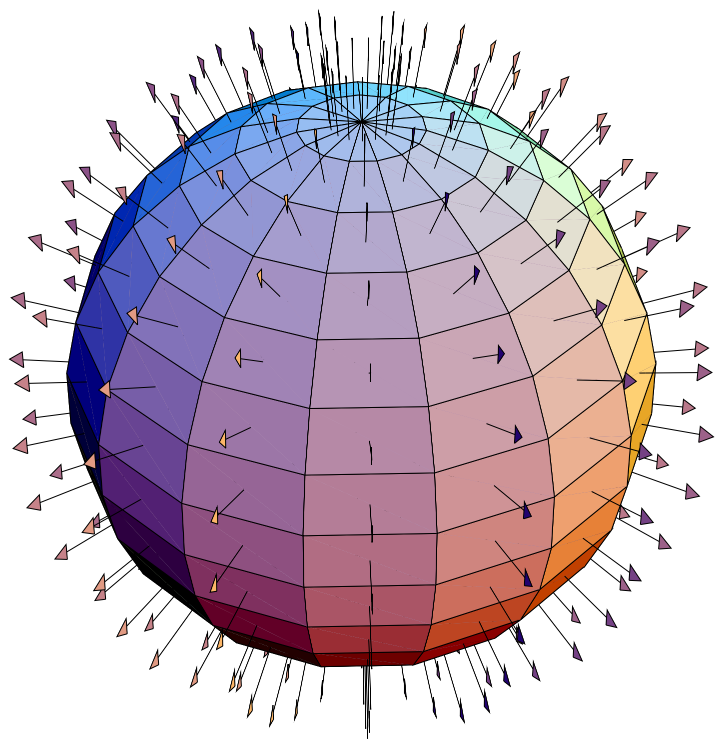
GAUGED SKYRMION MASSES ARE BOUNDED ABOVE

Normally, standard (ungauged) skyrmion masses are proportional to the coupling of the Skyrme term needed for stability, and so can grow to infinite magnitude with increasing coupling. Ultimately, they are an input. (Skyrme, 1961; Witten; Adkins, Nappi, and Witten, 1983).

In striking contrast, when skyrmions are gauged, their **masses are bounded above for any Skyrme coupling**.

Instead, they are of the order of monopole masses, $O(\frac{v}{g}) \sim \frac{M_W}{\alpha}$.

$\rightsquigarrow$ The actual coupling of the Skyrme term is not that crucial.



Effective theories of pseudoscalar mesons with global chiral symmetries, $SU(N)_L \times SU(N)_R$, (take $SU(2)_L \times SU(2)_R$; the axial ones are realized in the Nambu-Goldstone mode: $v = f_\pi \sim 93MeV$),

$$\frac{v^2}{4}\int d^4x \; \mathrm{Tr}\, \partial_\mu U^\dagger \partial^\mu U, \qquad \text{with} \qquad U = \exp\left(\frac{i\pi\cdot\tau}{v}\right) ,$$

support topologically stable meson field solutions.
Skyrme's spherical hedgehog Ansatz,

$$U = \exp\Big(if(r)\; \hat{\boldsymbol{x}}\cdot\boldsymbol{\tau}\Big) = \cos f(r) + i\hat{\boldsymbol{x}}\cdot\boldsymbol{\tau} \;\; \sin f(r).$$

The skyrmion chiral soliton solution is a "knot of pion fields": it carries a winding — a topological charge ($\Pi_3(SU(N))$), which emulates baryon number,

$$\frac{1}{24\pi^2}\int d^3x \quad \epsilon^{ijk}\mathrm{Tr}(U^\dagger\partial_i U\; U^\dagger\partial_j U\; U^\dagger\partial_k U) = \frac{1}{2\pi}\int dr\partial_r(2f - \sin 2f) \; .$$

$\rightsquigarrow$ Skyrmions provide an effective model of QCD baryons, multi-baryon nuclei, "pentaquarks", decay matrix elements, etc.

• Need for a harder **Skyrme term,** $\kappa^2\ \mathrm{Tr}[\partial_\mu U^\dagger, \partial_\nu U]^2$, to stabilize the core against collapse.
$\rightsquigarrow$ Sensitivity to the coupling strength of this term: the mass of the skyrmion is proportional to the square-root of the coupling coefficient κ^2 of the Skyrme term.

• This term has to be input, by hand, or to be somehow justified as emerging from a long-distance effective Lagrangian description of a more complicated system, e.g., from some shorter distance scale physics in QCD: You only get what you put in.

$$
\begin{aligned}
E(0,v,\kappa) &= \frac{v^2}{2}\int_0^\infty dr\,\left(r^2 f'^2 + 2\sin^2 f + \kappa^2\frac{\sin^4 f}{r^2} + 2\kappa^2\, f'^2 \sin^2 f\right) \\
&= \frac{v^2}{2}\int_0^\infty dr\,\left((rf' - \kappa\frac{\sin^2 f}{r})^2 + 2\sin^2 f\,(1-\kappa f')^2 + 6\kappa f' \sin^2 f\right).
\end{aligned}
$$

Scaling of κ into r: $\rightsquigarrow$ all activity occurs at scales of $r = O(\kappa)$,

$$\rightsquigarrow \quad \boxed{E \propto \kappa v^2}.$$

The first two terms in the integrand are positive semi-definite. The last one is a total divergence, $12\kappa\pi^2 r^2\times$ the topological (Chern-Simons) baryon density,

$$\frac{3\kappa}{2}\,\partial_r(2f - \sin 2f) = \frac{\kappa r^2}{2}\epsilon^{ijk}\mathrm{Tr}(U^\dagger\partial_i U\ U^\dagger\partial_j U\ U^\dagger\partial_k U)\ .$$

$\rightsquigarrow$ its contribution to the energy is

$$E_{topological} = \frac{3v^2\kappa}{4}\Big(2f(\infty) - 2f(0) - \sin 2f(\infty) + \sin 2f(0)\Big) = \frac{3\pi}{2}\,\kappa v^2.$$

This is a Bogomol'ny **topological lower bound**. However, it cannot be saturated, as it does in the BPS monopole case. (There are no self-dual chiral fields. Saturation would require both squares to vanish, both $f' = 1/\kappa$, and $\kappa \sin f = r$, which is impossible. $E \sim 1.2314 E_{top}$.)

This lower bound melts away for vanishing Skyrme term $\kappa = 0$, and blows up for $\kappa \to \infty$.

What happens in the presence of GAUGE INTERACTIONS?

• Perturbative gauge interactions on baryons (skyrmions), such as EW interactions.

• Skyrmions/baryons interact with the ρ-meson (Igarashi et al), which has an effective description as a gauge field of isospin.

• "Deconstruction": Pure Yang-Mills theories in higher dimensions that undergo compactification to $D = 4$, yield gauged chiral Lagrangians, in which the skyrmion matches higher dimensional topological objects: instantonic monopoles (Hill, Ramond).

• Gauged skyrmions in technibaryon decay (Rubakov).

For simplicity, take $SU(2)_L \times SU(2)_R$ with gauged diagonal $SU(2)_V$,

$$4\pi E(g,v,\kappa) \equiv \frac{1}{2}\int d^3x \,\mathrm{Tr} F_{ij}F_{ij} + \frac{v^2}{4}\int d^3x (\mathrm{Tr}[D_j,U^\dagger][D_j,U] + \kappa^2 \mathrm{Tr}\left([[D_j,U^\dagger],[D_i,U]]\right)^2),$$

with the simplest unit-winding Skyrme-Wu-Yang spherically symmetric hedgehog Ansatz,

$$A_i = \frac{a(r)-1}{gr}\,\epsilon_{ijk}\frac{\tau^j}{2}\hat{x}^k, \qquad U = \exp\Big(if(r)\hat{\boldsymbol{x}}\cdot\boldsymbol{\tau}\Big) = \cos f(r) + i\hat{\boldsymbol{x}}\cdot\boldsymbol{\tau}\,\sin f(r).$$

Rescale r and κ in units of gv:

$$E(g,v,\kappa)=\frac{v}{g}\int_0^\infty dr\left(4a'^2+\frac{2(a^2-1)^2}{r^2}+\frac{r^2f'^2}{2}+a^2\sin^2 f+\kappa^2a^4\frac{\sin^4 f}{2r^2}+\kappa^2\,a^2f'^2\sin^2 f\right),$$

BC $\quad a(0)=1,\quad a(\infty)=0;\quad f(0)=0,\quad f(\infty)=\pi.$

- Two-scale problem $E(g,v,\kappa)$ manifestly **monotonic in the Skyrme coupling strength** κ, because $\partial/\partial\kappa$ is positive semidefinite — all implicit dependence of the fields on κ vanishes on-shell (by use of the eqns of motion), and is thus irrelevant, as in the case of the monopole mass varying as a function of the Higgs mass.

- Also has a lower, topological bound (Brihaye et al),

$$\boxed{E_{topological}>\frac{2\pi v}{\sqrt{g^2+(\frac{4}{3\kappa v})^2}}}\ .$$

As $\kappa\to\infty$, κ-dependence drops out of the lower bound, of $O(v/g)$, BPS-monopolic. $\rightsquigarrow$ Once gauging is switched on, $1/g^2$ and κ^2 behave analogously to resistors in parallel: as κ blows up, it becomes irrelevant, leaving the scale to be set by g.

The lower bound results from rewriting the energy as

$$\begin{aligned}
E(g,v,\kappa) &= \frac{v}{g}\int_0^\infty dr\Bigg(\Big(4a'^2+\frac{(3\kappa/4)^2}{1+(3\kappa/4)^2}\frac{a^2\sin^2(2f)}{4}\Big)+\frac{(3\kappa/4)^2}{1+(3\kappa/4)^2}a^2\sin^4 f+a^2\sin^2 f\Big(\frac{1}{1+(3\kappa/4)^2}+\kappa^2\,f'^2\Big)\\
&+\frac{1}{2}\Big(\frac{r^2f'^2}{1+(3\kappa/4)^2}+\kappa^2a^4\frac{\sin^4 f}{r^2}\Big)+\frac{1}{2}\Big(\frac{(3\kappa/4)^2}{1+(3\kappa/4)^2}r^2f'^2+\frac{4(a^2-1)^2}{r^2}\Big)\Bigg)\\
&= \frac{v}{g}\int_0^\infty dr\Bigg(\Big(2a'+\frac{(3\kappa/4)}{\sqrt{1+(3\kappa/4)^2}}\frac{a\sin(2f)}{2}\Big)^2+\frac{(3\kappa/4)^2}{1+(3\kappa/4)^2}\,a^2\sin^4 f\\
&+a^2\sin^2 f\Big(\frac{1}{\sqrt{1+(3\kappa/4)^2}}-\kappa\,f'\Big)^2+\frac{1}{2}\Big(\frac{rf'}{\sqrt{1+(3\kappa/4)^2}}-\kappa a^2\frac{\sin^2 f}{r}\Big)^2\\
&+\frac{1}{2}\Big(rf'\frac{(3\kappa/4)}{\sqrt{1+(3\kappa/4)^2}}+\frac{2(a^2-1)}{r}\Big)^2\Bigg)+\frac{3\kappa v}{4g\sqrt{1+(3\kappa/4)^2}}\int_0^\infty dr\,\partial_r(2f-a^2\sin 2f)\\
&> \frac{2\pi v}{g\sqrt{1+(4/3\kappa)^2}}\ .
\end{aligned}$$

• The highest value for this lower bound, $2\pi v/g$, holds for $\kappa \to \infty$. Will show the actual energy is roughly twice this, in that limit, $\leadsto$ an **upper bound for the gauged skyrmion**, roughly twice this highest lower bound.

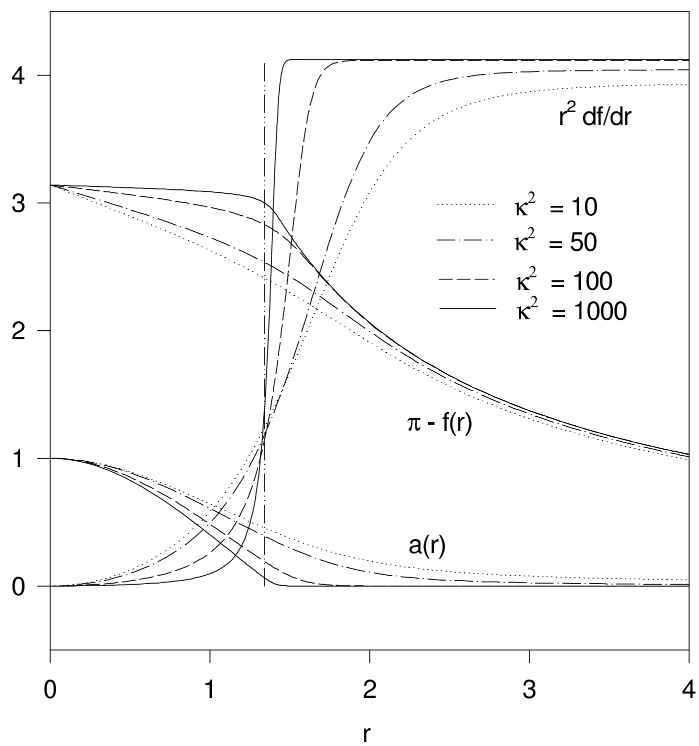
The Euler-Lagrange equations are

$$a'' + \frac{a(1-a^2)}{r^2} - \frac{a}{4}\sin^2 f - \frac{\kappa^2 a}{4} f'^2 \sin^2 f - \frac{\kappa^2 a^3}{4r^2}\sin^4 f = 0,$$

$$(r^2+2\kappa^2 a^2 \sin^2 f) f'' + 2r f' + 4\kappa^2 a a' f' \sin^2 f + \kappa^2 a^2 f'^2 \sin(2f) - a^2 \sin(2f) - \frac{2\kappa^2 a^4}{r^2}\sin^3 f \cos f = 0,$$

with BCs:

$$a(0) = 1, \qquad a(\infty) = 0; \qquad f(0) = 0, \qquad f(\infty) = \pi.$$



Numerically, for increasing κ^2, **the scales of the two variables a and f resolve** as in Born-Oppenheimer problems, and the monopole in the large Higgs mass limit. a asymptotes faster than f, which continues to evolve slowly after a has decayed to 0 by some value $r = R$.

For large $\kappa^2 \sim 1000$, $R \sim 1.34$, very weakly dependent on κ .

Abrupt transition of f' from null to $\pi R/r^2$ behavior.

In the proximate interval, $[0, R]$,

$$f(r) \sim 0,$$
$$a'' + \frac{a(1-a^2)}{r^2} \sim 0,$$

$\rightsquigarrow$ the Wu-Yang equation for pure Yang-Mills. (NB By itself, this is scale invariant: the actual scale R is set through interaction with f. The range of a would spread out, left to itself, but the Skyrme term disfavors overlap of a with f. In the limit, it forces a to attenuate inside the core, before f builds up at R.) $\rightsquigarrow$ $a(r)$ is an attenuating function which reaches $a(R) \sim 0$.

In the distant interval, $[R, \infty)$, $a \sim 0$,

$$\partial_r(r^2 f') \sim 0, \qquad \rightsquigarrow \qquad f \sim \pi\left(1 - \frac{R}{r}\right).$$

$\rightsquigarrow$ For $\kappa \to \infty$, dependence on κ dies out: For solutions, the coefficient of κ^2 in the energy collapses,

$$\frac{dE}{d(\kappa^2)} = \frac{\partial E}{\partial(\kappa^2)} = \int_0^\infty dr \left(a^2 f'^2 \sin^2 f + \frac{a^4 \sin^4 f}{2r^2}\right) \sim 1.62\,\kappa^{-5/2} \to 0.$$

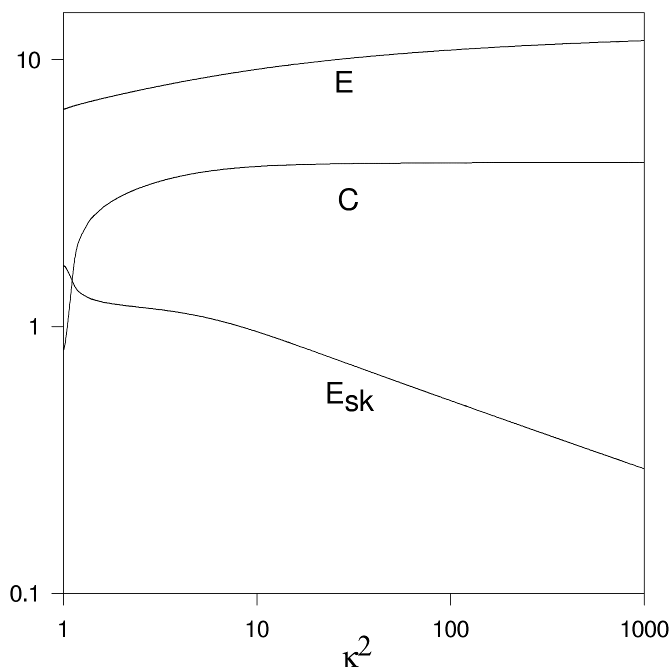
Asymptotically (tip of the hat to G Dunne!), activity is apparently dominated by the scale $r \sim \sqrt{\kappa}$ — quite unlike the characteristic scale of the ungauged skyrmion activity, $r \sim \kappa$.

Since E is monotonic in κ, an **upper bound** results in this limit,

$$\boxed{E(g, v, \kappa) \leq E(g, v, \infty) \sim 12.95\,\frac{v}{g}}\,.$$

• This upper bound is merely $\frac{g}{v}E(g, v, \infty) \sim 2.06 \times 2\pi$, where 2π is the above highest lower bound.

• In effect, the mass of the gauged skyrmion varies from 0 to 12.95 $\frac{v}{g}$, as the Skyrme term ranges from 0 to ∞ strength. Near zero, the Skyrme coupling κ sets the mass scale, but for large couplings the scale is set by the "monopole mass" scale v/g.



E in units of v/g, for increasing κ^2. The upper bound is at $E(g,v,\infty) \sim 12.95$. Further, the coefficient C of the leading tail of $f = \pi - C/r + O(1/r^2)$ for large r, tending to πR in the limit $\kappa \to \infty$. (Contrast to the $\pi - 2.16/r^2$ asymptoting of the conventional ungauged skyrmion.)
Also E_{sk}, the Skyrme term (the last two terms in E); decays like $E_{sk} \sim 1.62\kappa^{-1/2}$, subdominant to the contributions of the Wu-Yang and the conventional chiral terms (the leading four terms in E).

For $\frac{g}{v}E(g,v,\infty) \sim 12.95$, the subleading behavior is

$$E(g,v,\kappa) = E(g,v,\infty) - 6.68\,\frac{v}{g}\,\kappa^{-1/2} + O(\kappa^{-1}).$$

A cross-check: this expansion in $\kappa^{-1/2}$ around $\kappa^{-1/2} = 0$ is seen to be numerically consistent with

$$E_{sk} = \kappa^2 \frac{dE}{d(\kappa^2)} = -\frac{1}{4}\kappa^{-1/2}\frac{dE}{d(\kappa^{-1/2})}\ .$$

WHAT HAVE WE LEARNED?

The mass of the skyrmion increases monotonically from zero with the Skyrme coupling, but does not go to infinity as it would for ungauged skyrmions. Instead, the mass stabilizes to an upper bound, whose scale, $O(v/g)$, is "monopolic" — it is set by the gauge coupling and the characteristic spontaneous symmetry breaking scale.
This limit conforms to the masses of magnetic monopoles, which likewise do not vary much above the minimal BPS values.

BUT WHY?

The monopole is a tangle of gauge, Goldstone, and Higgs fields; as the mass of the Higgs field is taken to infinity with fixed VEV, the Higgs serves only to enforce boundary conditions. The monopole in this limit ends up made purely of gauge fields (a Higgsless monopole—Vinciarelli), with a mass of $O(M_W/\alpha)$.

Analogously, the gauged skyrmion consists of gauge fields and Higgs field skyrmions. But, in the interaction with very heavy would-be skyrmions, the last two terms in that system, the "Skyrme term" (viz., E_{sk}), become decreasingly relevant in the energy. Thus, what would have been the infinitely massive skyrmion largely enforces boundary conditions at R.
The leading two terms in the energy (the gauge, or Wu-Yang, part) scale as $1/s$ with $r \rightarrow sr$, and, left to themselves, favor a spread-out integrand to maximize s. The next two terms (the chiral action terms) scale as s, and favor core-shrinking. But the last two terms (the Skyrme terms) also scale as $1/s$ and oppose this. For large κ, the heavy lifting is done by the gauge part, and stabilizes the core to $\sim R$, constraining the gauge field within this range.

$\rightsquigarrow$ The mass of the skyrmion ends up of the order characteristic of monopole configurations, $O(v/g)$, superficially (only) oblivious of the Skyrme coupling.

UTILITY?

No need to worry about $\kappa \gg \frac{1}{gv}$ in estimates!