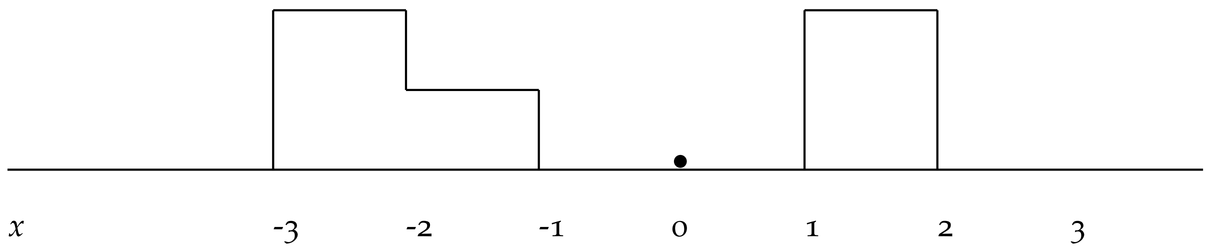
Homework Exercises for Quantum Mechanics in Phase Space

Q1: When does a WF vanish? To see where the WF $f(x_0, p_0)$ vanishes or not, for a given wavefunction $\psi(x)$ with *bounded support* (i.e. vanishing outside a finite region in x),

► Pick a point x_0 and reflect $\psi(x) = \psi(x_0 + (x - x_0))$ across x_0 to $\psi(x_0 - (x - x_0)) = \psi(2x_0 - x)$.

► See if the overlap of these two distributions is nontrivial or not, to get $f(x_0, p) \neq 0$ or $= 0$.

Now consider the *schematic* (unrealistic) real $\psi(x)$:

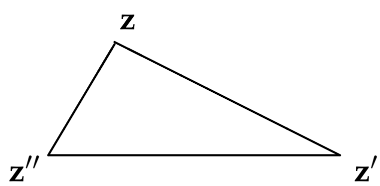


$\looparrowright$ Is $f(x_0 = -2, p) = 0$?

$\looparrowright$ Is $f(x_0 = 3, p) = 0$?

$\looparrowright$ Is $f(x_0 = 0, p) = 0$?

Q2: Define phase-space points $\mathbf{z} \equiv (x, p)$, etc. Consider



$$h(\mathbf{z}) \equiv f(\mathbf{z}) \star g(\mathbf{z}) = \int d\mathbf{z}' d\mathbf{z}'' f(\mathbf{z}') g(\mathbf{z}'') \;\; e^{k(\mathbf{z},\mathbf{z}',\mathbf{z}'')}.$$

What is $k(\mathbf{z}, \mathbf{z}', \mathbf{z}'')$? Is it related to the area of the triangle $\triangle(\mathbf{z}, \mathbf{z}', \mathbf{z}'')$? How?

Q3: Prove Lagrange's representation of the shift operator, $e^{a\partial_x}\, f(x) = f(x+a)$, possibly using the Fourier representation, or else action on monomials x^n. Now, evaluate $e_\star^{ax} \star e_\star^{bp}$. Also, evaluate $\delta(x) \star \delta(p)$.

Q4: Evaluate $e^{ax+bp} \star e^{cx+dp}$. Evaluate $(\delta(x)\ \delta(p)) \star (\delta(x)\ \delta(p))$.

Q5: Evaluate $G(x,p) \equiv e_\star^{ax\star p}$. Hint: Show $G \star x \propto x \star G$; find the proportionality constant; solve the first order differential equation in ∂_p...; impose B.C.

Q6: Prove the associativity of $\star$. (Hint: Baker integral rep. (13) or Q2).)

Q7: Prove footnote f. (Hint: Baker integral rep.)

Q8: Weyl-order $x^3 p^2$, ie, find its Weyl map. How many terms are there? can you find an equivalent re-expression with fewer terms, and no explicit $\hbar$s, using Heisenberg's commutation relation?

Q9: The two-state flip-flop, Exercise 0.2 at the end of Section 0.8: Evaluate just the simplest WF "snapshot" $f(x,0)$ as the WF rotates by $p=0$.

Q10: Derive eqn (73) for the Bargmann rep.

Q11: Exercise 0.7 In the Husimi representation (122) et seq., show f_H is normalized to 1. For the oscillator H_H given there, show

$$H_H \circledast f_{H\ n} = E_{H\ n}\, f_{H\ n}.$$

Is this differential eqn in z *simpler* than in the Wigner representation? (What order in z is it?) Hence, find the simple (un-normalized) f_Hs. (Alternatively, solve for U_H in $\hbar\partial_t U_H = iH_H \circledast U_H$, and thence read off these simple f_Hs.)

Q12: Prove footnote h: why is $x=p=0$ sufficient for the statement to hold for all x,p? More directly, show the following integral is always ≥ 0, for all x', p':

$$\int dydxdp\ \psi^*(x-y/2)\ \psi(x+y/2)\ e^{-iyp-(p-p')^2-(x-x')^2}\ ,$$

by first integrating out p, and then showing factorization into two mutually complex-conjugate integrals—or another proof of your choice!

Q13: Exercise 0.8

Q14: Check all formulas, like the Bopp shift (12), eqn (16), the captions of Figs 1 & 3; as well as (56) $\rightsquigarrow$ (58), (76) $\rightsquigarrow$ (79), (80) $\rightsquigarrow$ (85), (112) $\rightsquigarrow$ (111), and the Groenewold anomaly (129).

⋔ **Q15**, optional extra credit: Replicate in phase space Dirac's ladder $\star$-spectrum generation for the angular momentum functions, not operators, based on their MB $SO(3)$ algebra, $\{\{L_x, L_y\}\} = L_z$, etc.

Show the Casimir function $C \equiv \mathbf{L} \cdot \star \mathbf{L}$ is an invariant, $\{\{C, \mathbf{L}\}\} = 0$; and, for $L_\pm \equiv L_x \pm i L_y$,

$$C = L_+ \star L_- + L_z \star L_z - \hbar L_z \ ,$$

and

$$L_z \star L_+ - L_+ \star L_z = \hbar L_+ \ , \qquad \text{\& its C.C.,}$$

↷ so

$$\langle \mathbf{L} \cdot \star \mathbf{L} - L_z \star L_z \rangle = \langle L_x \star L_x + L_y \star L_y \rangle \geq 0 \ .$$

↷

Thus the $\star$-genvalues$/\hbar$, m, of L_z are integrally spaced, and moreover bounded in magnitude by any possible (non-negative!) lower bound l^2 of $C/\hbar^2$:

$$|m| \leq l \leq \sqrt{C}/\hbar \ .$$

↷ Thus there must be a "ground state" ("highest/lowest weight state") now for some *integer* (or 1/2-integer) l:

$$L_- \star f_{m=-l} = 0 \ ,$$

↻

$$L_+ \star L_- \star f_{-l} = 0 = (C - L_z \star L_z + \hbar L_z) \star f_{-l} \ ,$$

✓

$$\langle C \rangle = \hbar^2 l(l+1) \ .$$

Public URL Links

These are links to be enjoyed by all, and may be linked publically on anyone's web-page. The 2nd link is a perpetually updated version of the Overview used in these lectures. The first is a summary of the 1st lecture.

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