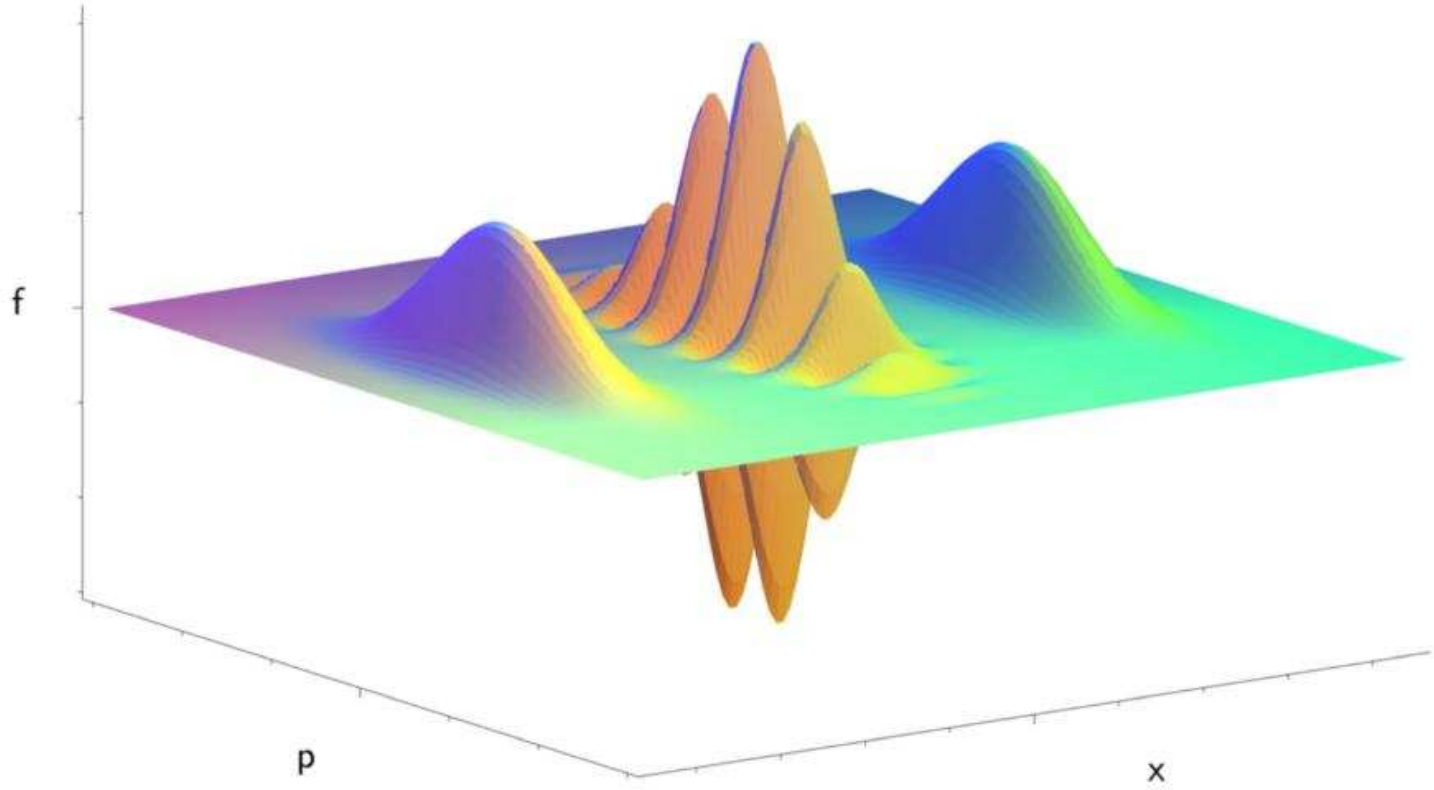
DOUBLE SLIT IN PHASE SPACE



The Wigner phase-space quasi-probability distribution function

QUANTUM MECHANICS LIVES AND WORKS IN PHASE-SPACE

A complete, autonomous formulation of QM based on the standard c-number variables x and p and their functions in phase-space, which compose through a special operation.

Three alternate paths to quantization:
1. Hilbert space (Heisenberg, Schrödinger, Dirac)
2. Path integrals (Dirac, Feynman)
3. Phase-space distribution function of Wigner (Wigner 1932; Groenewold 1946; Moyal 1949; Baker 1958; Fairlie 1964; ...)

$$f(x,p) = \frac{1}{2\pi}\int dy\, \psi^*\Big(x - \frac{\hbar}{2}y\Big)\, e^{-iyp}\psi\Big(x + \frac{\hbar}{2}y\Big).$$

A special representation of the density matrix (Weyl correspondence).

Useful in describing **quantum** transport/flows in phase space $\rightsquigarrow$ quantum optics; nuclear physics; study of decoherence (eg, quantum computing).

But also signal processing (time-frequency analysis); Intriguing mathematical structure of relevance to Lie Algebras, M-theory,...

Properties of $f(x,p) = \frac{1}{2\pi}\int dy\, \psi^*(x - \frac{\hbar}{2}y)\, e^{-iyp}\psi(x + \frac{\hbar}{2}y)$

Normalized, $$\int dpdx\, f(x,p) = 1\ .$$

- Real

- Bounded: $-\frac{2}{h} \leq f(x,p) \leq \frac{2}{h}$ (Cauchy-Schwarz Inequality) $\rightsquigarrow$ Cannot be a spike: Cannot be certain!

- p- or x-projection leads to marginal probability densities: A spacelike shadow $\int dp f(x,p) = \rho(x)$; **or else** a momentum-space shadow $\int dx f(x,p) = \sigma(p)$, resp; both positive semidefinite. But cannot be conditioned on each other. The uncertainty principle is fighting back $\rightsquigarrow$

- f can, and most often does go negative (Wigner). A hallmark of **quantum interference**. "Negative probability" (Bartlett; Moyal; Feynman; Bracken & Melloy).

Hiding through the uncertainty principle. Smoothing f by a filter of size larger than $\hbar$ (eg, convolving with phase-space Gaussian) results in a positive semidefinite function: it has been smeared or blurred to a classical distribution (de Bruijn, 1967). $\rightsquigarrow$ Negative areas are **small**!

When is a real $f(x,p)$ a bona-fide Wigner function?
When its Fourier transform L-R-factorizes
$\tilde{f}(x,y) = \int dp\, e^{ipy} f(x,p) = g_L^*(x-\hbar y/2)\, g_R(x+\hbar y/2)$,
($\frac{\partial^2 \ln \tilde{f}}{\partial(x-\hbar y/2)\partial(x+\hbar y/2)} = 0$), so $g_L = g_R$ from reality.

- Nevertheless, it **is** a distribution: it yields expectation values from phase-space c-number functions.

Given, in Weyl's association rule (1927), an operator $\mathbf{A}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ A(x,p)\exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x))$, the corresponding phase-space kernel function $A(x,p)$, obtained by $\mathbf{p}\mapsto p$, $\mathbf{x}\mapsto x$, yields that operator's expectation value:

$$\langle \mathbf{A}\rangle = \int dx dp\ f(x,p)\ A(x,p).$$

Dynamical evolution of f (Moyal):

Liouville's Thm, $\partial_t f + \{f,H\} = 0$, quantum generalizes to

$$\frac{\partial f}{\partial t} = \frac{H\star f - f\star H}{i\hbar}\ ,$$

based on the $\star$-product (Groenewold):

$$\boxed{\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)}},$$

the essentially unique one-parameter ($\hbar$) associative deformation of Poisson Brackets of classical mechanics. (viz. $\hbar\to 0$). (Isomorphism: $\mathbf{AB} = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp (A\star B)\exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x))$.)

Systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. But time-independent pure-state Wigner functions $\star$-commute with H.

However, they further obey a more powerful functional $\star$-genvalue equation (Fairlie, 1964):

$$H(x,p) \star f(x,p) = H\left(x + \frac{i\hbar}{2}\,\vec{\partial}_p,\; p - \frac{i\hbar}{2}\,\vec{\partial}_x\right)\, f(x,p)$$

$$= f(x,p) \star H(x,p) = E\; f(x,p)\;,$$

which amounts to a complete characterization of them:

For real functions $f(x,p)$, the Wigner form is equivalent to compliance with the $\star$-genvalue equation ($\Re$ and $\Im$ parts).
(Curtright, Fairlie, & Zachos, Phys Rev **D58** (1998) 025002)

$\Longrightarrow$ Projective orthogonality spectral properties

$$f \star H \star g = E_f\; f \star g = E_g\; f \star g.$$

For $E_g \neq E_f$, $\Longrightarrow$ $f \star g = 0.$

Precluding degeneracy, for $f = g$,

$$f \star H \star f = E_f\; f \star f = H \star f \star f,$$

$$\Longrightarrow \qquad f \star f \propto f.$$

fs $\star$-project onto their space.

$$\boxed{f_a \star f_b = \tfrac{1}{\hbar}\, \delta_{a,b}\, f_a.}$$

• The normalization matters (Takabayasi, 1954): despite linearity of the equations, it prevents superposition of solutions (this is not how QM interference works here!).

$$\int dpdx\; f \star g = \int dpdx\; fg,$$

so, for different $\star$-genfunctions,

$$\int dpdx\; fg = 0.$$

$\rightsquigarrow$ Negative values are a feature, not a liability. Quantum interference confined to "$\hbar$-small" regions.

NB $\rightsquigarrow$ $\displaystyle\int H(x,p) f(x,p)\; dxdp = E \int f\; dxdp\,.$

NB $\rightsquigarrow$ $\displaystyle\int f^2\;\; dxdp = \tfrac{1}{\hbar}\,.$

In general, $\leq 1/h$ $\rightsquigarrow$ quantum: fuzzy — classical: spiky.

• For any function, $\langle |g|^2 \rangle$ need not ≥ 0.

But $\langle g^* \star g \rangle \;\geq\; 0$ ($\rightsquigarrow$ the **uncertainty principle**, $\Delta x \Delta p \;\geq\; \hbar/2$ $\rightsquigarrow$ $(\Delta x)^2 + (\Delta p)^2 \geq \hbar$. Hides negative values).

▼ **Pf**

$$H(x,p)\star f(x,p)$$

$$\begin{aligned}
&= \frac{1}{2\pi}\left((p-i\frac{\hbar}{2}\,\vec{\partial}_x)^2/2m+V(x)\right)\int dy\; e^{-iy(p+i\frac{\hbar}{2}\overleftarrow{\partial}_x)}\psi^*(x-\frac{\hbar}{2}y)\;\psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\;\left((p-i\frac{\hbar}{2}\,\vec{\partial}_x)^2/2m+V(x+\frac{\hbar}{2}y)\right)e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\;\psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\; e^{-iyp}\left((i\,\vec{\partial}_y+i\frac{\hbar}{2}\,\vec{\partial}_x)^2/2m+V(x+\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\;\psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\; e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\;E\;\psi(x+\frac{\hbar}{2}y)=\\
&\qquad = E\;f(x,p);
\end{aligned}$$

• Action of the effective differential operators on ψ^* turns out to be null;

$$f\star H$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int dy\; e^{-iyp}\left(-(\vec{\partial}_y-\frac{\hbar}{2}\,\vec{\partial}_x)^2/2m+V(x-\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\;\psi(x+\frac{\hbar}{2}y)\\
&\qquad = E\;f(x,p).
\end{aligned}$$

Conversely, the pair of $\star$-eigenvalue equations dictate, for $f(x,p)=\displaystyle\int dy\; e^{-iyp}\tilde{f}(x,y)$,

$$\int dy\; e^{-iyp}\left(-\frac{1}{2m}(\vec{\partial}_y\pm\frac{\hbar}{2}\,\vec{\partial}_x)^2+V(x\pm\frac{\hbar}{2}y)-E\right)\tilde{f}(x,y)=0.$$

$\rightsquigarrow$ Real solutions of $H(x,p)\star f(x,p) = E\;f(x,p)$ $(=f(x,p)\star H(x,p))$ must be of the Wigner form,
$f=\displaystyle\int dy\; e^{-iyp}\psi^*(x-\tfrac{\hbar}{2}y)\psi(x+\tfrac{\hbar}{2}y)/2\pi$, (s.t. $\mathbf{H}\psi=E\psi$).

The wonderful truth (still!) sinking in: $\star$-multiplication of c-number phase-space functions is in complete isomorphism (Groenewold) to Hilbert-space operator algebra.

SIMPLE HARMONIC OSCILLATOR

Solve **directly** for $H = (p^2 + x^2)/2$
(with $\hbar = 1$, $m = 1$, $\omega = 1$):

$$\left((x + \frac{i}{2}\partial_p)^2 + (p - \frac{i}{2}\partial_x)^2 - 2E\right) f(x,p) = 0.$$

Mere PDEs! Imaginary part: $(x\partial_p - p\partial_x)f = 0. \rightsquigarrow$
f depends on only one variable, $z = 4H = 2(x^2 + p^2). \rightsquigarrow$

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - E\right) f(z) = 0.$$

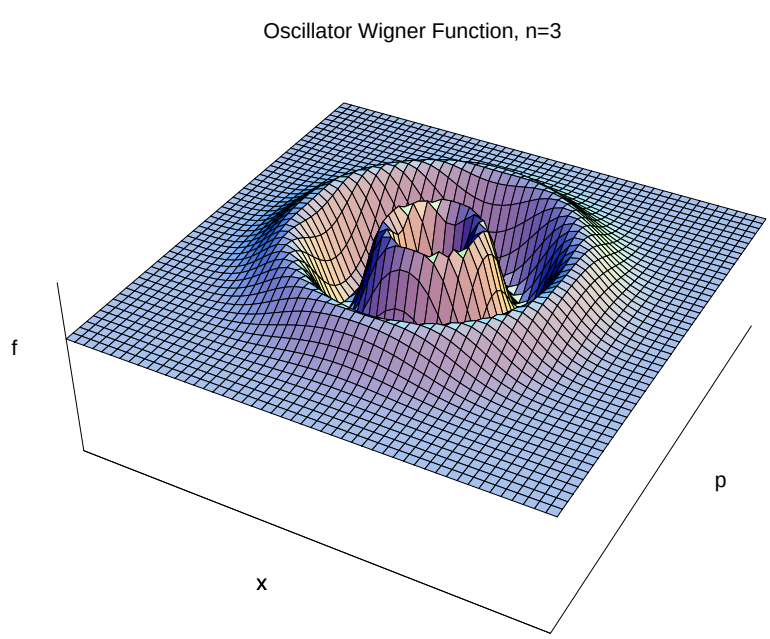
Set $f(z) = \exp(-z/2)L(z)$ $\Longrightarrow$ Laguerre's eqn

$$\left(z\partial_z^2 + (1-z)\partial_z + E - \frac{1}{2}\right) L(z) = 0.$$

Satisfied by Laguerre polynomials, $L_n = e^z\partial^n(e^{-z}z^n)/n!$, for $n = E - 1/2 = 0, 1, 2, \ldots \rightsquigarrow$ eigen-Wigner-functions are

$$f_n = \frac{(-1)^n}{\pi}\, e^{-2H}\, L_n(4H); \qquad L_0 = 1, \quad L_1 = 1 - 4H,$$

$L_2 = 8H^2 - 8H + 1, \ldots$ $\qquad \diamondsuit$ not positive definite.



$$\sum_n f_n = \tfrac{1}{2\pi} \ .$$

Dirac's Hamiltonian factorization for algebraic solution
carries through intact in $\star$ space:

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2} \,,$$

so define

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \qquad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip).$$

$$a \star a^\dagger - a^\dagger \star a = 1 \,.$$

$\star$-Fock vacuum:

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0 \,.$$

Associativity of the $\star$-product permits the customary ladder spectrum generation; $H \star f = f \star H$ $\star$-genstates:

$$f_n \propto (a^\dagger \star)^n \; f_0 \; (\star a)^n \,.$$

• real, like the Gaussian ground state;

$\rightsquigarrow$ left-right symmetric;

$\star$-orthogonal for different eigenvalues;

project to themselves, since the Gaussian ground state does, $f_0 \star f_0 \propto f_0$.

TIME EVOLUTION

Isomorphism to operator algebras $\rightsquigarrow$ associative combinatoric operations completely analogous to Hilbert space QM.

$\rightsquigarrow$ $\star$-unitary evolution operator, a "$\star$-exponential"

$$U_\star(x,p;t) = e_\star^{itH/\hbar} \equiv$$

$$1 + (it/\hbar)H(x,p) + \frac{(it/\hbar)^2}{2!}H \star H + \frac{(it/\hbar)^3}{3!}H \star H \star H + ...,$$

$$f(x,p;t) = U_\star^{-1}(x,p;t) \star f(x,p;0) \star U_\star(x,p;t).$$

NB Collapse to **classical** trajectories,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p\ ,$$

$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x \quad \Longrightarrow$$

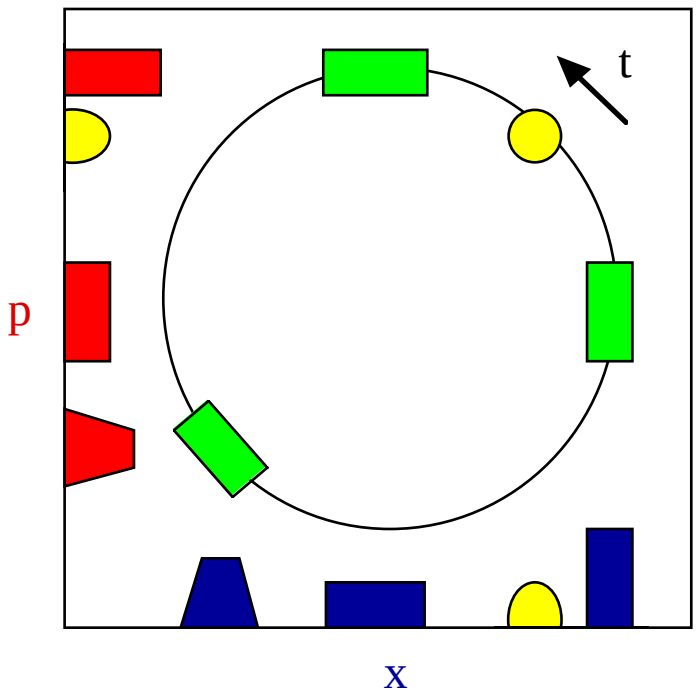
$$x(t) = x\cos t + p\sin t,$$

$$p(t) = p\cos t - x\sin t.$$

$\Longrightarrow$ For SHO the functional form of the Wigner function is preserved along classical phase-space trajectories (Groenewold, 1946):

$$f(x,p;t) = f(x\cos t - p\sin t, p\cos t + x\sin t; 0).$$

Any Wigner distribution rotates uniformly on the phase plane around the origin, essentially classically, even though it provides a complete quantum mechanical description.



NB In general, loss of simplicity upon integration in x (or p) to yield probability densities: the rotation induces shape variations of the oscillating probability density profile.

Only if (eg, coherent states) a Wigner function configuration has an additional axial $x-p$ symmetry around its *own* center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density.

THE WEYL CORRESPONDENCE BRIDGE

Weyl's correspondence map, by itself, merely provides **a change of representation between phase space and Hilbert space**
$\rightsquigarrow$ Mutual language to contrast classical to quantum mechanics on common footing.

$$\mathbf{A}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\; a(x,p)\exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x)),$$

Inverse map (Wigner):

$$a(x,p) = \frac{1}{2\pi}\int dy\; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\right|\mathbf{A}(\mathbf{x},\mathbf{p})\left|x-\frac{\hbar}{2}y\right\rangle\ .$$

PHASE SPACE		HILBERT SPACE
a	$\xrightarrow{Weyl}$	$\mathbf{A}$
$\text{quantum}\downarrow$		$\downarrow\text{quantum}$
$a\star b$	$\xrightarrow{Groenewold}$	$\mathbf{AB}$
$\text{classical } \hbar{=}0\downarrow$		$\downarrow\text{Bracken } \hbar{=}0$
ab	$\xrightarrow{Weyl}$	$\mathbf{A}\odot\mathbf{B}$

$\rightsquigarrow$ A plethora of choice-of-ordering quantum mechanics problems reduce to purely $\star$-product algebraic ones: varied deformations (ordering choices) can be surveyed systematically in phase space.
(Curtright & Zachos, New J Phys **4** (2002) 83.1-83.16 [hep-th/0205063])

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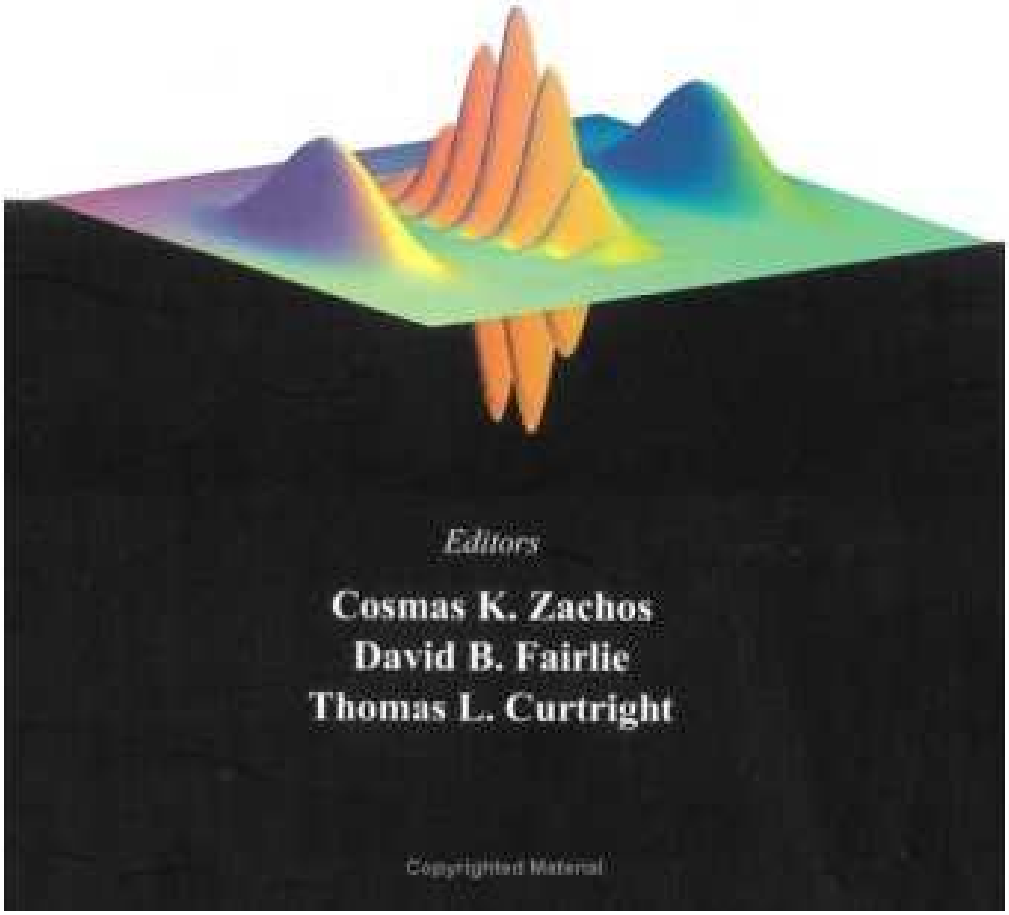
An Overview with Selected Papers



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