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PARADIGMS OF QUANTUM ALGEBRAS

Cosmas Zachos*

High Energy Physics Division, Argonne National Laboratory

Argonne, IL 60439-4815, USA (zachos@anlhep)

This is an informal overview of versions of quantum algebras which are currently finding applications in physics. Special attention is given to the quantum deformations of SU(2) and illustrations of general principles. It may serve as an eclectic introduction to the bibliography.

1. Introduction

Quantum Algebras, or QUE-(quantized universal enveloping)-algebras, are remarkable mathematical structures (noncommutative, noncocommutative Hopf algebras) which have been figuring in

i. 2-d solvable model S-matrices and solutions to their Yang-Baxter factorization equations [Kulish & Reshetikhin I, Sklyanin I, Faddeev et al., Jimbo I, Jimbo II, deVega, Itoyama, Ge, Wu, & Xue, Burroughs, Bernard & Leclair].

ii. Anisotropic spin chain hamiltonians [Pasquier & Saleur, Batchelor et al., Kulish & Sklyanin, Hou, Shi, Yang &Yue].

iii. 3-d Chern-Simons theory Wilson loops [Witten, Guadagnini et al., Majid & Soibelman, Siopsis]; topological QFTs [Majid II].

iv. Chiral vertices, fusion rules, and conformal blocks of RCFT [Alvarez-Gaumé et al., Moore & Reshetikhin, Gómez & Sierra, Itoyama & Sevrin, Furlan et al., Faddeev, Gawedzki, Alekseev & Shatasshvili, Ramírez et al.]; orbifolds [Bantay]; 2-d Liouville gravity [Gervais]; related applications of knot theory to physics [Kauffman, Saleur & Altschüler, Kauffman & Saleur].

v. q-strings and group-theoretic interpretation of q-hypergeometric functions [Romans, Masuda et al.].

vi. Nonstandard quantum statistics [Greenberg, Fivel]; squeezed light [Solomon & Katriel, Bužek, Celeghini et al. II].

vii. Heuristic phenomenology of deformed molecules and nuclei [Iwao, Raychev et al., Bonatsos et al., Celeghini et al. III, Chang et al.].

Quantum algebras become relevant in physics where the limits of applicability of Lie Algebras are stretched: they describe perturbations from some underlying symmetry structure,

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such as quantum corrections or anisotropies. They are currently being explored with a view to new applications in a broad range of contexts. There are several outstanding reviews of the subject, which also cover much of its interesting history and illuminate particular aspects of it [Drinfeld, Jimbo II, Faddeev et al., Manin I, Majid I, Takhtajan]. Here, I opt instead for a briefer, more eclectic, illustrative, and less historical introduction to these ideas. It is based on explicit prototypes, mostly addressing quantum deformations of SU(2), and techniques that may facilitate and encourage new applications.

2. Deformation of SU(2)

Consider the algebra of SU(2):

$$[j_x, j_y] = ij_z \qquad [j_y, j_z] = ij_x \qquad [j_z, j_x] = ij_y \ , \tag{2.1}$$

or, equivalently, for $j_x = (j_+ + j_-)/\sqrt{2}$, $j_y = -i(j_+ - j_-)/\sqrt{2}$, $j_z = j_0$,

$$[j_0, j_+] = j_+ \qquad [j_+, j_-] = j_0 \qquad [j_-, j_0] = j_- \ . \tag{2.2}$$

The Casimir invariant is

$$C \equiv j_x^2 + j_y^2 + j_z^2 = j_+j_- + j_-j_+ + j_0^2 = 2j_+j_- + j_0(j_0 - 1) \ . \tag{2.3}$$

Now suppose we mar the isotropy of this spherical expression by *deforming* it to:

$$C_q(j) \equiv j_+j_- + j_-j_+ + \frac{q+1/q}{2}(\frac{q^{j_0} - q^{-j_0}}{q - q^{-1}})^2 \ , \tag{2.4}$$

where the real or complex $q-1$ parameterizes the amount of anisotropy. q may be thought of as a phase, as in RCFT, or as $e^{\hbar}$, following historical development; in that case, the last term in C_q amounts to

$$\cosh\hbar \left(\frac{\sinh(\hbar j_0)}{\sinh(\hbar)}\right)^2,$$

which goes to the classical/isotropic limit as $\hbar \to 0$, i.e. $q \to 1$. Define, in general, the "*q-deformation of x*": $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$, so that $[x]_q \to x$ as $q \to 1$. Thus, the last term above amounts to

$$\frac{[2]_q}{2}\,[j_0]_q^2 \ .$$

Is most of the symmetry of the operator C_q gone (beyond the residual axial j_0)? It turns out in fact that it may be salvaged, provided the universal enveloping algebra of SU(2) is used in a suitable deformation. Define, with [Kulish & Reshetikhin I, Drinfeld, Jimbo I] new operators J_a which satisfy

$$[J_0, J_+] = J_+ \qquad [J_+, J_-] = \frac{1}{2}\,[2J_0]_q \qquad [J_-, J_0] = J_- \ , \tag{2.5}$$

which has (2.2) as its classical limit $q \to 1$. All of its generators now commute with C_q, written as

$$C_q(J) = 2J_+J_- + [J_0]_q[J_0 - 1]_q \ . \tag{2.6}$$

(2.5) is not a Lie algebra anymore, which forestalls its Lie-exponentiation to a group. It is a more general algebra: a Hopf algebra, which is to say that it is endowed with the following structures.

I. **Coproduct** Δ. This is an algebra homomorphism that corresponds to the composition of angular momenta, i.e. it specifies tensor (co)multiplication of representations. In the above example, it is [Sklyanin II, Jimbo I]:

$$\Delta_q(J_0) = J_0 \otimes \mathbb{1} + \mathbb{1} \otimes J_0 \qquad\qquad \Delta_q(J_\pm) = J_\pm \otimes q^{J_0} + q^{-J_0} \otimes J_\pm \ , \tag{2.7}$$

so that the $\Delta(J)$ satisfy the algebra (2.5), like a "total angular momentum". This coproduct is coassociative, but not cocommutative, since, defining the permutation map $\sigma(a \otimes b) \equiv b \otimes a$, you may note that $\sigma(\Delta_q) = \Delta_{1/q} \neq \Delta_q$. (This is an equally good coproduct, and still others are discussed below.) A given coproduct such as Δ_q determines the other two structures which, however, will not be crucial for this discussion:

II. **Counit** ϵ. This homomorphism reverses the effect of the above comultiplication :

$$(\epsilon \otimes \mathbb{1})\Delta(J_a) = 1 \otimes J_a, \qquad (\mathbb{1} \otimes \epsilon)\Delta(J_a) = J_a \otimes 1.$$

Here, it is $\epsilon(J_a) = 0$, $\epsilon(\mathbb{1}) = 1$.

III. **Antipode** S. This is a "hermitean transposition" algebra antihomomorphism, $S(J_a J_b) = S(J_b)S(J_a)$, s.t.

$$\sigma\Big(\Delta(S(J_a))\Big) = (S \otimes S)(\Delta(J_a)); \qquad m\Big((S \otimes \mathbb{1})\Delta(J_a)\Big) = m\Big((\mathbb{1} \otimes S)\Delta(J_a)\Big) = \epsilon(J_a),$$

given the multiplication map $m(a \otimes b) \equiv ab$ for spaces of matching dimension. Here, it is easy to check $S(J_\pm) = -q^{\pm 1} J_\pm$, $S(J_0) = -J_0$. Note the familiar classical limits of all of the above maps.

For generic q not equal to 1, the representation theory of this deformation, as detailed later, is in one-to-one correspondence with the representation theory of its classical limit, here the theory of angular momentum. Just as composing representations and taking functions of their Casimir invariants for SU(2) yields invariant hamiltonians, parallel comultiplications for SU(2)$_q$ provide a variety of invariants, out of which, for instance, important spin-chain hamiltonians have been identified to be invariant under SU(2)$_q$ [Pasquier & Saleur, Batchelor et al., Kulish & Sklyanin]. In (I) above, the alternative coproduct $\Delta_{1/q}$ was introduced, which is in fact equivalent to Δ_q via a similarity transformation: $\Delta_q = R_q \Delta_{1/q} R_q^{-1}$. This *universal R-matrix* of Drinfeld, with $R_q^{-1} = R_{1/q}$, leads to solutions of the Yang-Baxter equation, which is not reviewed here, as it is covered in detail in the reviews of [Jimbo II, Faddeev et al., Kosmann-Schwarzbach, Kirillov & Reshetikhin, deVega].

There are several alternate deformations of SU(2) available [Sklyanin I, Woronowicz, Witten, Fairlie I]. Each one has its distinctive invariants and representation theory, and all are related among themselves. To map them onto each other, one may first map them to this prototype deformation discussed, or to their common classical limit SU(2), as described next.

3. Deforming functionals and representation theory

The term "deformation" used above may, in fact, be made explicit [Curtright & Zachos]. Rewrite the classical invariant operator C, (2.3), as $j(j+1)$, where j is the formal operator $(\sqrt{1+4C} - 1)/2$. Then, by dint of the commutation relations of SU(2), the functionals

$$J_0 = Q_0(j_0) = j_0 \qquad J_+ = Q_+(g) = \sqrt{\frac{[j_0 + j]_q [j_0 - 1 - j]_q}{(j_0 + j)(j_0 - 1 - j)}}\ j_+ \qquad J_- = (Q_+(g))^\dagger \tag{3.1}$$

satisfy the commutation relations of SU(2)$_q$, (2.5). The maps $Q_\pm$ are functionals of all three SU(2) generators g : j_0, j_+, j_-, since they depend on the operator j. For nonreal q, the above conjugation is not hermitean, and taken *not* to complex-conjugate q.

Moreover, (2.6) now amounts to $[j]_q[j+1]_q$, i.e. a function of the classical invariant C. Conversely, for generic q (not a root of unity), one may further solve for j if only the J_a's are given:

$$2j+1 = \operatorname{arccosh}\Big((q+1/q+(q-1/q)^2C_q)/2\Big)\Big/\ln q\ . \tag{3.2}$$

Consequently, the functionals (3.1) are invertible, and their inverses Q^{-1} provide a realization of SU(2) in terms of quantum algebra generators, with the classical Casimir expressible as a function of the quantum one, C_q. These maps then provide realizations of each algebra in terms of the other. Thus, functions of C_q are also invariant under SU(2), while functions of C are also invariant under SU(2)$_q$[2]. As a result, these deforming maps specify the representation theory of each; e.g. when representations of SU(2) are substituted into (3.1), they yield the corresponding representations of SU(2)$_q$ of the same dimension. This underscores the general result that the representation theory of SU(2)$_q$ for generic q reduces to a "distorted echo" of the representation theory of SU(2) [Rosso II, Lusztig I, Vaksman & Soibelman]. Functionals of broadly analogous type have also appeared in [Jimbo I, Rosso I, Nomura, Macfarlane, Curtright I, Polychronakos I, Fairlie I].[3]

Having referred the representation theory of the QUE-algebra to the representation theory of SU(2), the above map links the respective composition laws for representations. It thus specifies a coproduct, which appears different from (2.7). The map-induced coproduct simply *classicizes the* SU(2)$_q$ *representations through the inverse maps* Q^{-1}*, it composes them at the classical level, and then it quantizes the answer through* Q. More specifically, in the classical addition of angular momenta, two parallel operators tensor-multiply to an operator satisfying the same SU(2) commutation relations; this operator is a reducible representation of SU(2), the reduction (and diagonalization of the cocasimir) effected by the Clebsch-Gordan operator $\mathcal{C}$:

$$\Delta(g) = \mathbb{1}\otimes g + g\otimes\mathbb{1} = \mathcal{C}(g_1\oplus g_2\oplus g_3\oplus ...)\mathcal{C}^{-1}\ . \tag{3.3}$$

Thus, the invertible map Q from SU(2) generators g to SU(2)$_q$ generators $G = Q(g)$ induces the following tensor coproduct of G's

$$Q(\Delta(g)) = Q\Big(\mathbb{1}\otimes Q^{-1}(G) + Q^{-1}(G)\otimes\mathbb{1}\Big)\ , \tag{3.4}$$

which obeys SU(2)$_q$ quommutations, since its argument obeys SU(2) [Curtright & Zachos, Polychronakos I]. Now the same Clebsch operator $\mathcal{C}$ will automatically also reduce the coproduct (3.4): $\mathcal{C}^{-1}Q(\Delta(g))\mathcal{C} = G_1\oplus G_2\oplus G_3\oplus ...$; this reduced coproduct is an equivalent one, since any similarity transformation on a coproduct will isomorphically produce an expression also satisfying the same algebra. The antipodes specified by the map (3.1) evidently amount to mere sign flips, just as in the classical algebra, and thus also appear different from (III); the resulting counit is likewise identical to the classical one.

The map-induced coproduct discussed is quite difficult to handle in some cases, and is not well-defined for q equal to a root of unity, as discussed later. How does it relate to the prototype Δ_q of the previous section? For generic q, that coproduct Δ_q reduces to a direct sum by the unitary q-Clebsch operators $\mathcal{C}_q$. Such coefficients are covered in [Vaksman, Kirillov & Reshetikhin,

[2] An extension to spin-chain hamiltonians, [Caldi et al.], contingent on their complete decomposition to irreducible blocks, uncovers SU(2) symmetry in anisotropic spin chains.

[3] Beyond the functionals sketched so far, various *non*invertible functionals are available which connect SU(1,1) with the centerless Virasoro algebra [Fairlie, Nuyts, & Zachos], or the classical SU(2) current algebra with SU(2)$_q$ [Itoyama & Sevrin], and others.

Pasquier, Nomura, Biedenharn, Ruegg, Koornwinder, Reshetikhin & Smirnov, Groza et al.], while the q-Wigner-Eckart theorem is worked out in [Biedenharn, Nomura, Bragiel]. Consequently,

$$Q(\Delta(g)) = \mathcal{C}\mathcal{C}_q^{-1}\, \Delta_q(G)\, \mathcal{C}_q\mathcal{C}^{-1} \equiv U_q^{-1}\, \Delta_q(G)\, U_q\ . \tag{3.5}$$

The induced comultiplication is thus related to (2.7) by a similarity transformation introduced in [Curtright, Ghandour, & Zachos], $\boldsymbol{U_q = \mathcal{C}_q\mathcal{C}^{-1}}$, the unitary operator that converts $\mathcal{C}$ to $\mathcal{C}_q$. Similarly, as already mentioned, Δ_q transforms to its double $\Delta_{1/q}$ through the operator $R_q = U_q U_{1/q}^{-1} = \mathcal{C}_q\mathcal{C}_{1/q}^{-1}$, which converts $\mathcal{C}_{1/q}$ to $\mathcal{C}_q$: $\Delta_q = R_q \Delta_{1/q} R_q^{-1}$. Some discussion of the broad equivalence class of coproducts is given in [Curtright, Ghandour, & Zachos]. The inverse functionals, in an unfolding of (3.3), moreover specify a non-cocommutative coproduct for classical SU(2) [Curtright II], which reduces by $\mathcal{C}_q$ instead of $\mathcal{C}$ and thus also transforms to the standard one (3.3) by the U matrix. For generic analysis see [Gerstenhaber & Schack]. Homological questions are addressed in [Feng & Tsygan].

It is worth illustrating the above general statements by substitution of unitary irreducible representations of SU(2) into formulas (3.1). The J_-'s follow from hermitean conjugation. The doublet representation (Pauli matrices):

$$j_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad j_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.6}$$

maps to itself for this deformation: $J_0 = j_0, \quad J_+ = j_+$. This is a special feature of the defining representation in this particular deformation. Note $C_q = 1 - [1/2]_q^2$. The **3**:

$$j_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad\qquad j_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

maps to $J_0 = j_0, \quad J_+ = \sqrt{(q+1/q)/2}\ j_+ = \sqrt{[2]_q/2}\ j_+$. The **4**:

$$j_0 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} \qquad\qquad j_+ = \begin{pmatrix} 0 & \sqrt{3/2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3/2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.8}$$

maps to

$$J_0 = j_0 \qquad\qquad J_+ = \begin{pmatrix} 0 & \sqrt{[3]_q/2} & 0 & 0 \\ 0 & 0 & [2]_q/\sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{[3]_q/2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

and so forth.

To illustrate coproducts (2.7,3.4), consider the $\mathbf{2}\otimes\mathbf{3}$ case. Classically, by (3.3) and (3.6-7),

$$\Delta(j_0) = \text{diag}\ (3/2, 1/2, -1/2, 1/2, -1/2, -3/2)\ ,$$

$$\Delta(j_+) = \begin{pmatrix} 0 & 1 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

reduce by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & -1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1/\sqrt{3} & 0 & -\sqrt{2/3} & 0 \\ 0 & 1/\sqrt{3} & 0 & \sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

to $\mathbf{4} \oplus \mathbf{2}$ blocks — the classical limit of (3.14) below. The same $\mathcal{C}$ also reduces $Q(\Delta(j_+))$.

However,

$$\Delta_q(J_+) = 1/\sqrt{2} \begin{pmatrix} 0 & \sqrt{[2]_q/q} & 0 & q & 0 & 0 \\ 0 & 0 & \sqrt{[2]_q/q} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/q \\ 0 & 0 & 0 & 0 & \sqrt{[2]_q q} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{[2]_q q} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.12}$$

reduces instead through

$$\mathcal{C}_q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{[2]_q/[3]_q q} & 0 & -q/\sqrt{[3]_q} & 0 & 0 \\ 0 & 0 & 1/q\sqrt{[3]_q} & 0 & -\sqrt{[2]_q q/[3]_q} & 0 \\ 0 & q/\sqrt{[3]_q} & 0 & \sqrt{[2]_q/[3]_q q} & 0 & 0 \\ 0 & 0 & \sqrt{[2]_q q/[3]_q} & 0 & 1/q\sqrt{[3]_q} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.13}$$

to $\mathbf{4} \oplus \mathbf{2}$ blocks,

$$\mathcal{C}_q^{-1}\Delta_q(J_+)\mathcal{C}_q = 1/\sqrt{2} \begin{pmatrix} 0 & \sqrt{[3]_q} & 0 & 0 & 0 & 0 \\ 0 & 0 & [2]_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{[3]_q} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.14}$$

Naturally, the q-cocasimir diagonalizes to $\mathcal{C}_q^{-1}\Big(2\Delta_q(J_+)\Delta_q(J_-) + [\Delta_q(J_0)][\Delta_q(J_0) - 1]\Big)\mathcal{C}_q =$ diag $([3/2][5/2],\ [3/2][5/2],\ [3/2][5/2],\ [1/2][3/2],\ [1/2][3/2],\ [3/2][5/2])$, which bears the expected functional relationship to its clasical limit. The reader ought to check all corresponding classical limits.

The two quantum coproducts are related by

$$U_q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c(q) & 0 & s(q) & 0 & 0 \\ 0 & 0 & c(1/q) & 0 & -s(1/q) & 0 \\ 0 & -s(q) & 0 & c(q) & 0 & 0 \\ 0 & 0 & s(1/q) & 0 & c(1/q) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$c(q)=\frac{\sqrt{2[2]_q/q+q}}{\sqrt{3[3]_q}}, \qquad s(q)=\frac{\sqrt{[2]_q/q-\sqrt{2}q}}{\sqrt{3[3]_q}}\ , \tag{3.15}$$

and therefore

$$R_q=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{[2]+1}{[3]} & 0 & (\frac{1}{q}-q)\frac{\sqrt{[2]}[3/2]}{[3]} & 0 & 0\\ 0 & 0 & \frac{[2]+1}{[3]} & 0 & (\frac{1}{q}-q)\frac{\sqrt{[2]}[3/2]}{[3]} & 0\\ 0 & (q-\frac{1}{q})\frac{\sqrt{[2]}[3/2]}{[3]} & 0 & \frac{[2]+1}{[3]} & 0 & 0\\ 0 & 0 & (q-\frac{1}{q})\frac{\sqrt{[2]}[3/2]}{[3]} & 0 & \frac{[2]+1}{[3]} & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$=(\frac{1}{q}-q)\frac{2[3/2]_q}{[3]_q}(J_+\otimes J_- - J_-\otimes J_+)+2(1-\frac{[2]_q+1}{[3]_q})J_0^2\otimes J_0^2+(1-\frac{[2]_q+1}{[3]_q})J_0\otimes J_0+\frac{[2]_q+1}{[3]_q}\,\mathbb{1}\otimes\mathbb{1}\,. \tag{3.16}$$

Dramatic new features emerge as q becomes an Nth root of unity, hence $[N/2]_q=0$, however [Lusztig II, Roche & Arnaudon, Alvarez-Gaumé et al., Saleur I, Ganchev & Petkova, Sun et al.], which is of special relevance to RCFT. Inspection of the deforming functionals (3.1) indicates that:

a. *The dimensionality of the irreducible representations is bounded above by N.* (The constraints are actually twice as stringent for even Ns, since the effective period is $N/2$—see the above references). $J_\pm$ become nilpotent, $J_\pm^N=0$,[4] which may be seen from the vanishing products $[j_0+j][j_0+j-1]....[j_0+j+2-N][j_0+j+1-N]$ resulting inside the square-roots of the Nth power of (3.1) through the identity $j_+f(j_0)=f(j_0-1)j_+$. Thus there is only a *finite* number of irreducible representations for SU(2)$_q$. Consequently, it is necessary that large irreps of SU(2) map to reducible representations of SU(2)$_q$, as the raising/lowering within a representation is interrupted by the zeros inside the square-roots of (3.1). For example, observe that $[3]_q=0=[3/2]_q$ for $q=\exp(2\pi i/3)$. The **4** representation J_+ now has only one nontrivial entry and $J_+^2=0$; the middle commutator in (2.5) breaks up, so the representation reduces: $\mathbf{4}=\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{1}$.

b. The invariant operator C_q *does not label representations uniquely anymore.* E.g. for odd N, the invariant for any representation of dimension $2j+1$ coincides with that of dimension $2j'+1\equiv nN-(2j+1)$, integer n, or dimension $2j+1+nN$. Such representations with identical Casimir operators can mix into *indecomposable* but not irreducible representations, provided the collective *q-dimension*, $\sum q^{2j_0}=[2j+1]+[2j'+1]$, of the composite representation vanishes [Pasquier & Saleur]. Pasquier & Saleur term such representations "type I". Full reduction fails by dint of the divergence of $\mathcal{C}_q$ [Curtright, Ghandour, & Zachos, Zachos][5]. For example, for $q=\exp(2\pi i/3)$ again, the nonunitary **6** of eq. (3.12) is reducible, but not

[4] More generally, [De Concini & Kac], such powers belong to the center of the algebra; in the representations discussed here, the center is null. For a nontrivial center, the ensuing periodic and semi-periodic irreps,[Date et al., Gómez et al., Arnaudon II], are labelled by three (two) complex parameters and lose correspondence to classical representations. Their coproduct may not only fail to decompose, as below, but it may even not intertwine via a universal R-matrix , in contrast to the irreps discussed here.

[5] This is also implicit in [Reshetikhin & Smirnov].

decomposable to a **4** and a **2**, as their collective q-dimension vanishes: $[4]+[2]=0$. Specifically, since $J_- = J_+^t$, the norm is $v \cdot v = v^t v$. The six states $a^t = (0,0,0,0,0,1)$, $d^t = (-1,0,0,0,0,0)$, $b \equiv J_+ a$, $c \equiv J_- d$, $b'^t = q\sqrt{2}(0,0,1,0,-i,0)$, $c'^t = q^2\sqrt{2}(0,i,0,-1,0,0)$, contain the doublet of zero-norm states b and c, which only transform to each other: $J_+ b = c/\sqrt{2}$, $J_- c = b/\sqrt{2}$, $J_- b = 0$, $J_+ c = 0$. However, as evident above, a and d are not singlets, and may, in turn, be reached from elsewhere: $J_- b' = a$, $J_+ c' = d$; $b \cdot b' = c \cdot c' = 1$, and $J_- c' = (b+b')/\sqrt{2}$, $J_+ b' = (c + c')/\sqrt{2}$. However, the r.h.s. of (3.14) decomposes completely, so divergence of $\mathcal{C}_q$ is necessary, and likewise of U_q, but not of R_q. The reader would profit from working out more examples so as to develop facility for applications.

c. If, in addition, unitarity is required, substantially more stringent constraints ensue on the allowed dimensionalities of the irreducible representations [Mezincescu & Nepomechie]. $SU(2)_q$ and $SU(1,1)_q$ are linked, as unitary representations of one are "antiunitary" ones ($J_+^\dagger = -J_-$) of the other and vice-versa. The dimensionalities of these unitary/antiunitary representations are given by Takahashi-Suzuki numbers, while there is also a class of irreducible representations of indefinite hermitean conjugation signature. (E.g. the **4** for $q = \exp(2\pi i/5)$. Again, the reader may wish to practice with (3.9)). Also see [Keller, Dobrev].

4. Other deformations of SU(2), and generalizations to other algebras

The deforming functionals exemplified above are by no means unique. Nonhermitean functionals are also found in [Jimbo I, Curtright & Zachos] and, in general, any nonsingular similarity transform of the functionals discussed will also do.

There is a number of interesting alternative deformations of $SU(2)_q$ which have arisen in several contexts, listed below:

i. The trigonometric limit of Sklyanin's elliptic deformation [Sklyanin I, Macfarlane]:

$$[S_0, S_3] = 0\,, \qquad [S_+, S_-] = 4S_0 S_3\,, \qquad [S_3, S_\pm] = \pm(S_0 S_\pm + S_\pm S_0)\,,$$

$$[S_0, S_\pm] = \pm(S_\pm S_3 + S_3 S_\pm)\tanh^2\eta\,, \quad \text{where} \quad S_0^2 - S_3^2 \tanh^2\eta = 4\sinh^2\eta\,, \tag{4.1}$$

with classical limit $\eta \to 0$ (upon rescaling of the generators), and an invariant

$$C_\eta = S_+ S_- + S_- S_+ + S_3^2\left(\frac{2\cosh 2\eta}{\cosh^2\eta}\right)\,. \tag{4.2}$$

(The original *full* elliptic deformation being

$$[S_3, S_0] = \frac{k\cosh\eta}{\cosh 2\eta}(S_+^2 - S_-^2)\,, \qquad [S_+, S_-] = 2\{S_0, S_3\}\,, \qquad [S_3, S_\pm] = \pm\{S_0, S_\pm\}\,,$$

$$[S_0, S_\pm] = \pm\frac{\sinh^2\eta - k^2}{\cosh^2\eta - k^2}\{S_\pm, S_3\} \pm \frac{k}{\cosh\eta\,(\cosh^2\eta - k^2)}\{S_3, S_\mp\}\,.$$

Its two quadratic invariants

$$K_0 = S_+S_- + S_-S_+ + 2S_3^2 + 2S_0^2, \qquad K_2 = S_+S_- + S_-S_+ + \frac{k}{\cosh\eta}(S_+^2 + S_-^2) + \frac{2\cosh 2\eta}{\cosh^2\eta}\,S_3^2,$$

combine to the above constraint and C_η in the trigonometric limit $k \to 0$.)

ii. Woronowicz's deformation [Woronowicz] has a linear r.h.s., but "quommutators" in lieu of commutators:

$$[V_0, V_+]_{s^2} \equiv s^2 V_0 V_+ - \frac{1}{s^2}V_+V_0 = V_+ \qquad\qquad [V_-, V_0]_{s^2} = V_-$$

$$[V_+, V_-]_{1/s} \equiv \frac{1}{s}V_+V_- - sV_-V_+ = V_0\,. \tag{4.3}$$

The invariant,

$$C_s = 2\Big(V_-V_+ + \frac{(1 - V_0(1 - 1/s^2))}{s(s - 1/s)^2}\Big)\Big/\sqrt{1 - (s^2 - 1/s^2)V_0}\,, \tag{4.4}$$

strictly commutes with the generators, i.e. $[C_s, V] = 0$. In the classical $s \to 1$ limit, this reduces to the operator (2.3) plus the divergent constant $(1-s)^{-2}/2 - 1/8$.

iii. Witten's first deformation [Witten]:

$$[E_0, E_+]_p \equiv pE_0E_+ - \frac{1}{p}E_+E_0 = E_+ \qquad [E_+, E_-] = E_0 - (p - \frac{1}{p})E_0^2 \qquad [E_-, E_0]_p = E_-\,. \tag{4.5}$$

The Casimir operator which commutes with all generators is:

$$C_p = \frac{1}{p}E_+E_- + pE_-E_+ + E_0^2 \qquad\qquad [C_p, E] = 0. \tag{4.6}$$

iv. Witten's second deformation [Witten]:

$$[W_0, W_+]_r \equiv rW_0W_+ - \frac{1}{r}W_+W_0 = W_+ \qquad [W_+, W_-]_{1/r^2} = W_0 \qquad [W_-, W_0]_r = W_-\,. \tag{4.7}$$

Observe the symmetry $W_0 \leftrightarrow -W_0$, $W_+ \leftrightarrow W_-$, $r \leftrightarrow 1/r$. For arbitrary functions f, it follows that

$$W_+f(W_0) = f(r^2W_0 - r)W_+ \;\text{ and }\; W_+f(W_-W_+) = f(W_+W_-)W_+ = f(r^4W_-W_+ + r^2W_0)W_+,$$

and their $+/-$ symmetric analogs. As a result, by virtue of

$$C_1 = 2W_-W_+ + \frac{2}{r^2(r + 1/r)}W_0(W_0 + r), \qquad C_2 = (1 - (r - 1/r)W_0)^2,$$

$$[C_i, W_\pm]_{r^{\pm 2}} = 0 \qquad\qquad [C_i, W_0] = 0\,, \tag{4.8}$$

a Casimir operator which commutes with all generators is:

$$C_r = C_1/C_2 \qquad\qquad [C_r, W] = 0. \tag{4.9}$$

Observe that (ii, iii, iv) have SU(2) as their $s = 1$, $p = 1$, and $r = 1$ limit, respectively, and SU(1,1) as their $s = -1$, $p = -1$, and $r = -1$ limit.

v. The two deformations (ii), (iv) are special limits of a 2-parameter generalization of [Fairlie I],

$$[I_0, I_+]_r = I_+ \qquad [I_+, I_-]_{1/s} = I_0 \qquad [I_-, I_0]_r = I_- \,, \tag{4.10}$$

upon $r \to s^2$, or $s \to r^2$, respectively. The corresponding invariant is

$$I_{r,s} = C_1/C_2 \,, \tag{4.11}$$

$$C_1 = 2I_-I_+ + \frac{2}{r-1/r}\Big(\frac{1}{s-1/s} - \frac{1-(r-1/r)I_0}{s-r^2/s}\Big)\,, \qquad C_2 = (1-(r-1/r)I_0)^{\ln s/\ln r}.$$

Linear combinations of C_1 and C_2 are equally acceptable in the numerator of the above invariant, which leads to the limit (4.9).

vi. The cyclically symmetric deformation [Odesskii, Fairlie I]:

$$qXY - q^{-1}YX = Z \qquad qYZ - q^{-1}ZY = X \qquad qZX - q^{-1}XZ = Y \,, \tag{4.12}$$

with a cubic invariant

$$\begin{aligned} C_q &= (q^3 + 2q^{-1})(XYZ + YZX + ZXY) - (q^{-3} + 2q)(XZY + ZYX + YXZ) = \\ &= \frac{2q^4 + 5 + 2q^{-4}}{(q-1/q)(q^2 - 1 + q^{-2})}\Big([X,Y]_Q, Z]_Q + [Y,Z]_Q, X]_Q + [Z,X]_Q, Y]_Q\Big), \end{aligned} \tag{4.13}$$

where $Q^2 \equiv (q^3 + 2q^{-1})/(q^{-3} + 2q)$. The Casimir invariant goes to the conventional one in the classical limit—the vanishing of the denominator of the coefficient exactly compensates for the collapse of the Q-determinant to the Jacobi identity [Zachos].

Deforming maps which map the representation theories (including the special limits q = roots of unity in the previous section) to that of each other, either directly, or via SU(2) are also available. E.g. consider (iv) above. A map to the prototype deformation (2.5) is [Curtright & Zachos]

$$W_0 = \frac{r^{-J_0}}{r+1/r}(r^{1+j}[J_0 - j]_r + r^{-1-j}[J_0 + j]_r) = \frac{1}{r-1/r}\Big(1 - \frac{r^{2j+1} + r^{-2j-1}}{r+1/r} r^{-2J_0}\Big)$$

$$W_+ = r^{-J_0}\sqrt{\frac{2r}{r+1/r}}\sqrt{\frac{[J_0+j]_r[J_0-1-j]_r}{[J_0+j]_q[J_0-1-j]_q}}\, J_+ \,, \tag{4.14}$$

for which the Casimir invariant (4.9) reduces to[6]

$$C_r = \frac{2\,[2j]_r[2j+2]_r}{r^2(r+1/r)(r^{2j+1} + r^{-2j-1})^2} \quad . \tag{4.15}$$

By virtue of (3.1), it is evident that (4.14) is identical with its $q = 1$ limit and also represents, in fact, a map from (2.2) to (4.7). Moreover, note the substantial simplification when $q = r$:

$$W_0 = \frac{1}{r-1/r}\Big(1 + \frac{[2j]_r - [2j+2]_r}{[2]_r}\, r^{-2J_0}\Big)\,, \qquad W_+ = r^{1/2 - J_0}\sqrt{\frac{2}{[2]_r}}\, J_+ \,, \tag{4.16}$$

which, e.g., allows a rapid inspection of the limit when r is a root of unity; as in the previous section, the zeros of $W_\pm$ dictate breakup of large representations and impose the same bounds on dimensionalities of irreps.

Analogous functionals exist in the literature for each of the above deformations:

[6]This amounts to (5.10) of [Witten], up to a factor of $2r^{-2}/[4]_r$.

i. [Roche & Arnaudon, Macfarlane];

ii. [Sudbery, Curtright & Zachos, Rosso I];

iii. [Nomura, Curtright & Zachos];

iv. [Curtright & Zachos];

v. [Curtright & Zachos];

vi. [Fairlie I, Zachos, Curtright II].

In this sense, these deformations are "equivalent" being all equivalent to SU(2). A trivial (cocommutative) coproduct is thereby always induced, as sketched in the previous section. Normally, invertibility is lost for $q =$ root of unity, but several of the direct maps among deformations survive, and map the respective modular representations discussed in the previous section to each other, as exemplified for (iv). [Gerstenhaber & Schack] discuss equivalence more generally, as well as obstructions and the cocycle structure of such deformations.

The above generalizes beyond SU(2) to the other Lie Algebras [Reshetikhin, De Concini & Kac, Dobrev]. Specifics include:

a. $\mathrm{SU}(1{,}1)_q$ mentioned already, [Bernard & Leclair]: unitary irreps for generic q discussed by [Masuda et al., Kulish & Damashinksy, Klimyk et al.]; [Zachos] probes modular representations.

b. $\mathrm{SU}(N)_q$ and their affine (Kac-Moody) extensions [Drinfeld, Jimbo I, Reshetikhin & Semenov, Woronowicz]; [Ueno et al.] investigate the representation theory; [Arnaudon I] constructs all periodic and flat representations of $\mathrm{SU}(3)_q$; [Date et al., Arnaudon & Chakrabarti] study the periodic representations; symmetric representations also discussed in [Sun & Fu, Polychronakos I] via q-oscillator realizations described below. Further see [Soibelman]. [Bernard & Leclair] apply the Yangian affine extension to non-local symmetries.

   There exists an intriguing deformation of the Moyal algebra [Fairlie II]:

$$q^{\mathbf{n}\times\mathbf{m}} J_{\mathbf{m}} J_{\mathbf{n}} - q^{\mathbf{m}\times\mathbf{n}} J_{\mathbf{n}} J_{\mathbf{m}} = \left(\omega^{\mathbf{m}\times\mathbf{n}/2} - \omega^{\mathbf{n}\times\mathbf{m}/2}\right) J_{\mathbf{m}+\mathbf{n}} + \mathbf{a}\cdot\mathbf{m}\,\delta_{\mathbf{m}+\mathbf{n},0} \qquad (4.17)$$

   which holds promise for practical applications. The indices are 2-vectors with integer entries, $\mathbf{m} = (m_1, m_2)$, $\mathbf{m}\times\mathbf{n} = m_1 n_2 - m_2 n_1$, and $\mathbf{a}$ is an arbitrary 2-vector characterizing the center. The classical limit is the Moyal algebra [Fairlie & Zachos I], which identifies with a maximally graded basis of SU(N) for $\omega = e^{2\pi i/N}$ (the natural generalization of the Pauli matrices to $N > 2$): in this cyclotomic case, all indices identify mod N, and consequently there are only N^2 different J's. For nontrivial q, in the limit $N \to \infty$, this provides the generalization of (vi) to $\mathrm{SU}(\infty)_q$, upon proper rescaling of the generators. This is the quantum version of the Poisson Bracket. (Contrast to [Levendorskii & Soibelman].)

c. $\mathrm{SO}(N)_q$, $\mathrm{Sp}(N)_q$, in [Reshetikhin, Jing, Nakashima, Kashiwara]. Also see [Gavrilik & Klimyk]

d. The exceptional algebras have been approached in [Reshetikhin, Koh & Ma, Ma].

e. The graded algebras: $\text{Osp}(2|1)_q$ is detailed in [Kulish & Reshetikhin II, Devchand, Saleur II, Bracken et al., Curtright & Ghandour]; $\text{Osp}(2|2)_q$ in [Deguchi, Fuji, & Ito]; $\text{Osp}(N|2)_q$ in [Chaichian, Kulish, & Lukierski]; $\text{Gl}(N|1)_q$ in [Palev & Tolstoy]. $\text{Sl}(N|M)_q$ in [Chaichian & Kulish, Bracken et al.]; $\text{Gl}(N|M)_q$ in [Zhang].

f. A candidate for the q-deformation of the Virasoro algebra has been proposed [Curtright & Zachos] and investigated [Chaichian, Kulish, & Lukierski, Polychronakos II, Narganes-Quijano], but, in the absence of a coproduct, it is not known to be a Hopf algebra. The operators

$$Z_m = x^{-m}\,\frac{r^{2x\partial}-1}{r-1/r}$$

satisfy the deformation of the centerless Virasoro algebra

$$[Z_m, Z_n]_{r^{n-m}} = [m-n]_r\; Z_{m+n}\;. \tag{4.18}$$

The operators Z_1, Z_0, Z_{-1} comprise the $\text{SU}(1{,}1)_q$ analog of (iv). In the limit $r \to 1$, Z_m yields the standard Virasoro realization $x^{1-m}\partial$. It does not appear possible to introduce a satisfactory center [Polychronakos II].

g. It is possible to map $\text{SU}(2)_q$ to the q-Heisenberg algebra, which is ultimately traceable to unpublished work of Heisenberg through [Rampacher et al.]. Consider the following formal contraction of [Chaichian & Ellinas] and [Ng] (contrast to [Celeghini et al. I]; also see [Yan]):

$$b \equiv q^{J_0} J_- \sqrt{2(q-1/q)}\;, \qquad\qquad b^\dagger \equiv J_+ q^{J_0}\sqrt{2(q-1/q)} \tag{4.19}$$

so that

$$[J_0, b^\dagger] = b^\dagger\;, \qquad\qquad [b, J_0] = b \tag{4.20}$$

and hence

$$bb^\dagger - q^2 b^\dagger b = 1 - q^{4J_0}. \tag{4.21}$$

The last term on the r.h.s. vanishes e.g. for $|q| > 1$, $J_0 << 0$ in an infinite-dimensional representation (Schwinger's contraction), to yield the q-oscillator algebra [Cigler, Kuryshkin, Jannussis et al., Macfarlane, Biedenharn, Kulish & Damashinksy]

$$bb^\dagger - q^2 b^\dagger b = 1. \tag{4.22}$$

The conventional realization for this algebra is $b^\dagger = x$ and $b = D_{q^2}$, where D_q is the quantum derivative, i.e. the slope of the chord to the graph of a function between x and qx:

$$D_q f(x) \equiv \frac{f(qx) - f(x)}{x(q-1)}\;.$$

If the number operator

$$N \equiv \ln\Big(1 + (q^2-1)b^\dagger b\Big)/\ln q^2$$

is introduced [Macfarlane], hermitean for real q, s.t.

$$[N, b^\dagger] = b^\dagger\;, \qquad\qquad [N, b] = -b\;, \tag{4.23}$$

this q-oscillator algebra can be mapped to the alternate form

$$\alpha = q^{-N} b, \qquad \alpha^\dagger = b^\dagger q^{-N} \qquad \Longrightarrow \qquad [\alpha, \alpha^\dagger] = q^{-2N} \tag{4.24}$$

with $\alpha^\dagger\alpha = (1-q^{-2N})/(1-q^{-2})$; or else the hybrid form

$$a = q^{-N/2} b, \qquad a^\dagger = b^\dagger q^{-N/2} \qquad \Longrightarrow \qquad aa^\dagger - qa^\dagger a = q^{-N}\;, \tag{4.25}$$

with $a^\dagger a = [N]_q$. These q-Heisenberg algebras provide a natural language for nonstandard quantum statistics [Greenberg, Fivel].

In general, maps with exponentials of number operators serve to rewrite quommutator algebras such as those listed at the beginning of this section, (ii-vi), to commutator ones with a more complicated right-hand-side; for instance [Curtright II],

$$[H_0, H_\pm] = \pm H_\pm \qquad [H_+, H_-] = \frac{s^{1+2H_0}(1 - r^{-2H_0})}{r - 1/r} \tag{4.26}$$

deforms to (4.10) of (v) via

$$I_+ = s^{-H_0} H_+ , \qquad I_- = H_- s^{-H_0}, \qquad I_0 = \frac{1 - r^{-2H_0}}{r - 1/r} . \tag{4.27}$$

Deforming functionals for the q-Heisenberg algebra (4.25) are [Cigler, Kuryshkin, Jannussis et al., Polychronakos I]:

$$a^\dagger = \sqrt{\frac{[N]}{N}} A^\dagger, \qquad a = A \sqrt{\frac{[N]}{N}} , \tag{4.28}$$

where the classical oscillator algebra is

$$[A, A^\dagger] = 1, \qquad N = A^\dagger A , \tag{4.29}$$

consistent with the above.

The fermionic analog of the above bosonic oscillators,

$$\{\psi, \psi\} = 0 = \{\psi^\dagger, \psi^\dagger\}$$
$$\psi\psi^\dagger + q\psi^\dagger\psi = q^N, \tag{4.30}$$

is, in fact, trivial [Floreanini et al., Jing & Xu], since the q-fermions coincide with the undeformed fermions, i.e. the deformer is the identity: For undeformed fermion oscillators, $N = \psi^\dagger\psi$, so that $N^2 = N$, and, consequently, $q^N = 1 - N + qN$. Moreover, note that, independently of the value of the parameter q, which appears to alter the interchange statistics, the deformed fermions still exclude themselves, whereas the deformed bosons still do not.

The Jordan-Schwinger realization [Biedenharn & Louck] of, e.g., SU(N) generators via classical oscillators:

$$J^a = A_i^\dagger T^a_{ij} A_j,$$

where T^a_{ij} are fundamental representation matrices, serves to produce symmetric representations. Substitution of q-oscillators a_i for the classical A_i's yields a realization of SU$(N)_q$ [Sun & Fu, Polychronakos I, Biedenharn, Macfarlane, Ruegg, Nomura] which produces "q-symmetric" representations, and affords a practical glimpse into their structure.[7] A more general treatment of oscillator and spinor representations for all unexceptional q-algebras is available in [Hayashi].

The foregoing Jordan-Schwinger bilinear constructions have the various q-oscillators commute with each other trivially. One may, however, demand a more general structure, subject to the associativity constraints discussed in the next session. What emerges for N oscillators is an $N(N+1)/2$-parameter q-Heisenberg algebra [Fairlie & Zachos II], which then yields a deformation of GL(N) with $(N-1)(N-2)/2$ parameters. A more standard multiparameter structure which is covariant under the group GL(N)$_q$ and connects to the quantum hyperplane is developed in [Pusz & Woronowicz, Wess & Zumino, Manin II, Sudbery, Schirrmacher, Vokos].

[7] Also see the treatment of SU(1,1)$_q$ by [Kulish & Damashinksy], SL(2|1)$_q$ by [Chaichian & Kulish], and Osp(1|2N)$_q$ by [Floreanini et al.]. Fermionic q-oscillators produce "q-antisymmetric" representations [Floreanini et al.]. q-symmetrizers are discussed in [Pusz & Woronowicz, Wybourne et al.].

5. Miscellany and outlook

All the structures discussed in the above survey are, of course, associative (and coassociative). To confirm associativity in quommutator algebras, one must verify the "braid-Jacobi" (Yang-Baxter) relations. This consists [Manin II, Polychronakos II] of using the quommutator algebra to permute the operators in a trilinear product $J_1 J_2 J_3$ in two alternate ways:

$$\begin{tikzcd} & 132 \arrow[r] & 312 \arrow[dr] & \\ 123 \arrow[ur] \arrow[dr] & & & 321\ . \\ & 213 \arrow[r] & 231 \arrow[ur] & \end{tikzcd}$$

Comparing coefficients of the resulting terms of each order in 321 reached by the two pathways indicates whether associativity is a direct consequence of the algebra, as is the case in all algebras listed above, with the exception of the q-Virasoro candidate in (f) of the previous section, for which extra quadratic constraint relations result (and so are necessary for associativity).

The link of QUE-algebras to q-groups differs from the standard connection between Lie Algebras and Lie Groups. [Faddeev et al., Rosso I, Nomura][8] provide realizations of q-groups in terms of $SU(2)_q$ generators with the proper q-group relations. Nevertheless, the classical limit of these realizations is virtually trivial, in that it does not determine the conventional "exponential composition" of the Lie algebra — a workable q-deformation of that exponentiation for all representations is still unavailable. Conversely, [Masuda et al.] construct the QUE-algebra generators out of q-group elements. Perhaps illustrative, a particular realization I find in the spirit of that setting is the following.

Faddeev and Takhtajan's $SL_q(2)$ q-group [Manin I] of uni-q-modular 2×2 q-matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is specified by the following component relations:

$$qab = ba \qquad qcd = dc \qquad bc = cb$$
$$qac = ca \qquad qbd = db \qquad ad - bc/q = da - qbc = 1\ . \tag{5.1}$$

The following formal functionals of q-group entries:

$$J_0 = \ln \sqrt{bc}/\ln q\ , \qquad J_+ = \frac{i\,2^{-1/2}}{q-1/q}\, a\, \sqrt{1+\frac{1}{qbc}}\ , \qquad J_- = \frac{i\,2^{-1/2}}{q-1/q} \sqrt{1+\frac{1}{qbc}}\, d \tag{5.2}$$

reproduce the $SU(2)_q$ algebra commutation relations:

$$[J_0, J_+] = J_+ \qquad [J_-, J_0] = J_-\ ,$$
$$[J_+, J_-] = \frac{-2^{-1}}{(q-1/q)^2}\Big((1+q/bc)ad - (1+1/qbc)da\Big) = [2J_0]_q/2\ , \tag{5.3}$$

[8]Also see [Woronowicz, Vokos, Zumino, & Wess, Sudbery, Wess & Zumino] for the connection to noncommutative differential geometry. In particular, [Sudbery] connects group and algebra via duality.

so this is a nonhermitean[9] realization of the q-algebra in terms of the q-group $SL_q(2)$ elements. Conversely,

$$bc = q^{2J_0} , \qquad a = -i(q-1/q)J_+\sqrt{\frac{2}{1+q^{-2J_0-1}}} , \qquad d = -i(q-1/q)\sqrt{\frac{2}{1+q^{-2J_0-1}}}J_- . \quad (5.4)$$

Applications. Selected applications were listed at the beginning of this overview[10]. Given the wealth of deformations, invariants, and ready reference to classical SU(2), further applications with intriguing prospects may include: construction of q-solvable potentials and use of q-algebras for spectrum-generation; construction of spin-chain hamiltonians with the alternate deformations listed as their degeneracy algebras; and several other opportunities to perturb beyond some underlying Lie algebraic structure.

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[9] $C_q = -(\sqrt{q} - 1/\sqrt{q})^{-2}$.

[10] For less compelling applications, see [Iwao, Raychev et al., Chaichian, Ellinas, & Kulish].

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