RECENT HIGHER ORDER QCD RESULTS: The QCD beta function at 4-loops and the first moment of g_1 in O(α_s^3)

work done with:
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- The method used for these 4-loop calculations
- The result for the 4-loop beta function
- The result for $\int_0^1 dx\, g_1(x, Q^2)$ in the order α_s^3

THE FOUR LOOP β-FUNCTION IN QUANTUM CHROMODYNAMICS

S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, hep-ph/9701390

The β-function is known to 3-loops (in the $\overline{\text{MS}}$ scheme:)

$$
\begin{aligned}
\frac{\partial a_s}{\partial \ln \mu^2} &\equiv \beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6), \\
\beta_0 &= 11 - \frac{2}{3} n_f \\
\beta_1 &= 102 - \frac{38}{3} n_f \\
\beta_2 &= \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \\
\beta_3 &= \, ?
\end{aligned}
$$

Tarasov, Vladimirov, Zharkov (1980)

$$\left(a_s = \frac{\alpha_s}{4\pi}\right)$$

- The β-function is not a physical object but it enters in all results obtained within perturbative QCD

Is the calculation of the 4-loop β-function in QCD feasible ?

(to illustrate the order of magnitude:)

Classes:

Renormalization constants:

$\tilde{Z}_1$	Z_3	$\tilde{Z}_3$

$\neq$ 4-loop diagrams:

30, 834	18, 794	2, 334

$$g_0 = \frac{\tilde{Z}_1}{\tilde{Z}_3} \frac{1}{\sqrt{Z_3}}\, g\,, \qquad \beta_{\mathrm{MS}} = \left(\frac{D-4}{2}\right) \frac{g_0^2}{\left(\frac{\partial g_0^2}{\partial g^2}\right)}$$

Alternative sets of diagrams

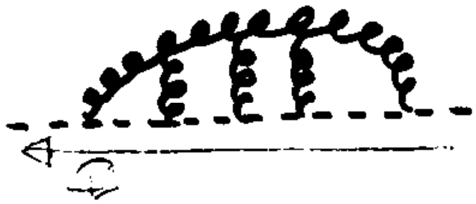
$\approx$ 150.000 diagrams

$$g_0 = Z_1\,(Z_3)^{-3/2}\, g$$

$\approx$ 1/2 million dias

$$g_0 = Z_4\left(Z_1\sqrt{Z_3}\right)^{-1} g$$

Basic method to obtain the overall UV divergence of all Feynman diagrams in dimensional regularization

 − (UV subdivergences)

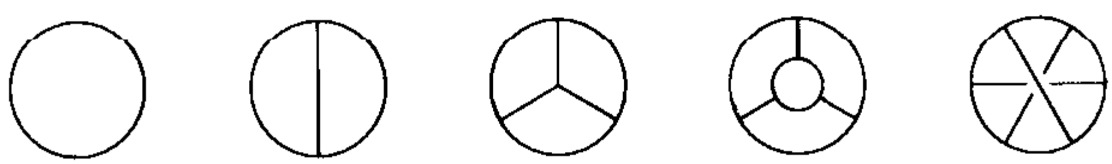
$$= Q^2\,[\text{overall UV divergence}] \; + \; \text{UV finite part}$$

- Introduce a "dummy" mass M in all the propagators as an infrared cutoff, e.g.:

$$\frac{1}{p^2}\left(g^{\mu\nu} + \xi\frac{p^\mu p^\nu}{p^2}\right) \rightarrow \frac{1}{p^2+M^2}\left(g^{\mu\nu} + \xi\frac{p^\mu p^\nu}{p^2+M^2}\right)$$

- Make Taylor expansions in external momenta to simplify the integrals. $M \neq 0$ **prevents introducing infrared divergences.**

Note: there are only few integration topologies after Taylor expansions:

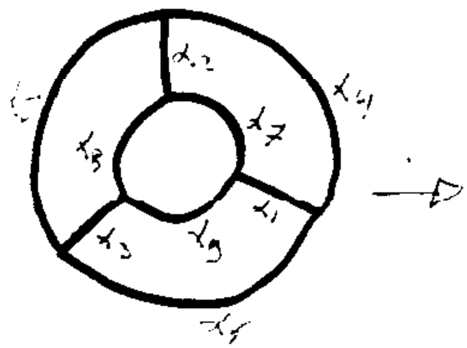


- Set the M to zero only after the integrations.

Note: $M \rightarrow 0$ is well defined since all non-local terms $\log^i(M^2/\mu^2)$ cancel in the overall UV-divergence.

(This cancellation is seen explicitly)

One can express the vacuum bubble integrals of 4-loops, i.e.



$$\int\int\int\int \frac{1}{[p_1^2 + M^2]^{\alpha_1}[p_2^2 + M^2]^{\alpha_2}} \cdots$$

α_1, α_2, $\cdots$, α_{10} integers

in terms of only a few ($\approx$ 10) "primitive" 4-loop integrals using recursion relations (obtained via:)

$$0 = \int d^D P_i \frac{\partial}{\partial P_i^\mu} P_j^\mu \begin{bmatrix} \text{some integrand} \\ \text{containing } P_i \end{bmatrix}$$

— The pole parts of the "primitive" 4-loop bubbles can be expressed using INFRARED REARRANGEMENT in terms of 3-loop massless propagator-type integrals.

The Result:

$$\frac{\partial a_s}{\partial \ln \mu^2} \equiv \beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6),$$

$$\begin{aligned}
\beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_F n_f \\
\beta_1 &= \frac{34}{3} C_A^2 - 4 C_F T_F n_f - \frac{20}{3} C_A T_F n_f \\
\beta_2 &= \frac{2857}{54} C_A^3 + 2 C_F^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f \\
&\quad - \frac{1415}{27} C_A^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2 \\
\beta_3 &= C_A^4 \left(\frac{150653}{486} - \frac{44}{9} \zeta_3 \right) + C_A^3 T_F n_f \left(-\frac{39143}{81} + \frac{136}{3} \zeta_3 \right) \\
&\quad + C_A^2 C_F T_F n_f \left(\frac{7073}{243} - \frac{656}{9} \zeta_3 \right) + C_A C_F^2 T_F n_f \left(-\frac{4204}{27} + \frac{352}{9} \zeta_3 \right) \\
&\quad + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2 \left(\frac{7930}{81} + \frac{224}{9} \zeta_3 \right) + C_F^2 T_F^2 n_f^2 \left(\frac{1352}{27} - \frac{704}{9} \zeta_3 \right) \\
&\quad + C_A C_F T_F^2 n_f^2 \left(\frac{17152}{243} + \frac{448}{9} \zeta_3 \right) + \frac{424}{243} C_A T_F^3 n_f^3 + \frac{1232}{243} C_F T_F^3 n_f^3 \\
&\quad + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left(-\frac{80}{9} + \frac{704}{3} \zeta_3 \right) + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left(\frac{512}{9} - \frac{1664}{3} \zeta_3 \right) \\
&\quad + n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \left(-\frac{704}{9} + \frac{512}{3} \zeta_3 \right)
\end{aligned}$$

with:

$$\begin{aligned}
d_F^{abcd} &= \frac{1}{6} \mathrm{Tr} \left[T^a T^b T^c T^d + T^a T^b T^d T^c + T^a T^c T^b T^d \right. \\
&\qquad \left. + T^a T^c T^d T^b + T^a T^d T^b T^c + T^a T^d T^c T^b \right] \\
d_A^{abcd} &= \frac{1}{6} \mathrm{Tr} \left[C^a C^b C^c C^d + C^a C^b C^d C^c + C^a C^c C^b C^d \right. \\
&\qquad \left. + C^a C^c C^d C^b + C^a C^d C^b C^c + C^a C^d C^c C^b \right]
\end{aligned}$$

$$T^a T^b - T^b T^a = i f^{abc} T^c$$

$$[C^a]_{bc} \equiv -i f^{abc}$$

Checks of the result for β_3:

- Check of gauge invariance

$$\text{gluon propagator} \rightarrow \frac{i}{q^2 + i\epsilon}\left(-g^{\mu\nu} + (1-\xi)\frac{q^\mu q^\nu}{q^2 + i\epsilon}\right)$$

- Comparison with known 4-loop QED result

  $C_A = 0$, $d_A^{abcd} = 0$, $C_F = 1$, $T_F = 1$, $d_F^{abcd} = 1$, $N_A = 1$

  Gorishny Kataev Larin Surguladze 1991

- Comparison of n_f^3 terms with large n_f result

  known in all orders of α_s: J.A. Gracey, Phys.Lett.B373 (1996) 178.

- The tensor d and the constants ζ_3 should cancel in N=1 supersymmetric gauge theory

  remark of D.R.T. Jones: $C_F = C_A = T_F, n_f = 1/2$, $d_F = d_A$

- Many constants (like ζ_4, ζ_5) cancel in final result

- Comparison with Padé estimates

  J. Ellis, M. Karliner, M.A. Samuel, hep-ph/9612202

$$\beta_3^{\text{pade}} = 23600 - 6400 n_f + 350 n_f^2 + 1.5 n_f^3$$

$$\beta_3 \approx 29243 - 6946 n_f + 405 n_f^2 + 1.5 n_f^3$$

THE α_s^3 APPROXIMATION OF QCD TO THE ELLIS-JAFFE SUM RULE

S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, hep-ph/9702435

The hadronic tensor in terms of **structure functions:**

$$
\begin{aligned}
W_{\mu\nu}(p,q) &= \frac{1}{4\pi}\int d^4 z e^{iq\cdot z}\langle \mathrm{nucl}|J_\mu(z)J_\nu(0)|\ \mathrm{nucl}\rangle \\
&= e_{\mu\nu}\frac{1}{2x}F_L(x,q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,q^2) \\
e_{\mu\nu} &= \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \\
d_{\mu\nu} &= \left(-g_{\mu\nu} \quad p_\mu p_\nu\frac{4x^2}{q^2} \quad (p_\mu q_\nu + p_\nu q_\mu)\frac{2x}{q^2}\right),
\end{aligned}
$$

In the present work: J_μ is the electromagnetic quark current

If the initial state hadron is **polarized**, use additional structure functions g_1 and g_2

$$
\begin{aligned}
W_{\mu\nu}^{\text{spin dep.}}(x,q^2) = W_{\mu\nu}^{\text{spin average}}(x,q^2) \\
+i\varepsilon_{\mu\nu\rho\sigma}\,q_\rho\left(\frac{S_\sigma}{P\cdot q}\,g_1(x,q^2) + \frac{S_\sigma P\cdot q - P_\sigma P\cdot S}{(P\cdot q)^2}g_2(x,q^2)\right)
\end{aligned}
$$

S is the spin vector of the initial state hadron

In the Bjorken limit $[Q^2 \to \infty, 0 < x \leq 1]$: relate moments of $W_{\mu\nu}$ via a dispersion relation to the Operator Product Expansion of $J_\mu(z)J_\nu(0)$

The Ellis-Jaffe sum-rule is expressed as

$$\int_0^1 dx g_1^{p(n)}(x, Q^2) = C^{ns}(1, a_s(Q^2))(\pm \tfrac{1}{12}|g_A| + \tfrac{1}{36}a_8) + C^s(1, a_s(Q^2))\, \tfrac{1}{9}\, a_0(Q^2)$$

g_A, a_8, a_0 are proton matrix elements of axial currents:

$$\begin{aligned} |g_A| s_\sigma &= 2\langle p, s|J_\sigma^{5,3}|p, s\rangle = (\Delta u - \Delta d)s_\sigma, \\ a_8 s_\sigma &= 2\sqrt{3}\langle p, s|J_\sigma^{5,8}|p, s\rangle = (\Delta u + \Delta d - 2\Delta s)s_\sigma, \\ a_0(\mu^2)s_\sigma &= \langle p, s|J_\sigma^5|p, s\rangle = (\Delta u + \Delta d + \Delta s)s_\sigma = \Delta\Sigma(\mu^2)s_\sigma. \end{aligned}$$

Where:

$J_\sigma^{5,a}(x) = \overline{\psi}\gamma_\sigma\gamma_5 t^a \psi(x)$ (Non Singlet axial current)

$J_\sigma^5(x) = \sum_{i=1}^{n_f} \overline{\psi_i}\gamma_\sigma\gamma_5\psi_i(x)$ (Singlet axial current)

Note:

$|g_A| = 1.2601 \pm 0.0025$: the constant of the *neutron* beta-decay.
$a_8 = 0.579 \pm 0.025$: the constant of *hyperon* decays.

The relevant Operator Product Expansion is

$$i \int dz e^{iqz} T\{J_\mu(z)J_\nu(0)\} \overset{Q^2\to\infty}{=} \epsilon_{\mu\nu\rho\sigma}\frac{q_\rho}{q^2}\left[\sum_a C^a\left(\tfrac{\mu^2}{Q^2}, a_s(\mu^2)\right) J_\sigma^{5,a}(0) + C^s\left(\tfrac{\mu^2}{Q^2}, a_s(\mu^2)\right) J_\sigma^5(0)\right] + \cdots$$

The renormalization group equations give the Q^2-dependence via:

$$a_0(Q^2) = \exp\left(\int_{a_s(\mu^2)}^{a_s(Q^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)} \right) a_0(\mu^2)$$

$\beta(a_s)$ is QCD beta-function, γ^s is the singlet anomalous dimension:

$$\frac{d}{d\ln\mu^2}[J^5_\mu]_R = \gamma^{\mathrm{s}}[J^5_\mu]_R$$

We use the renormalization group invariant matrix element:

$$\hat{a}_0 = \exp\left(- \int^{a_s(\mu^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)} \right) a_0(\mu^2)$$

Since $\hat{a}_0$ is the renormalization group invariant it should be considered as a physical constant on the same ground as the constants g_A and a_8.

Note: Non singlet axial current $J^{5,a}_\mu$ is conserved $\rightarrow$
the corresponding non-singlet anomalous dimension vanishes:

$$\rightarrow \frac{d}{d\ln\mu^2}[J^{5,a}_\mu]_R = 0$$

So, the non singlet matrix elements g_A and a_8 are renormalization group invariant (as physical quantities should be)

Known higher order contributions (for $n_f = 3$:)

$$\int_0^1 dx g_1^{p(n)} = \left[1 - \left(\frac{\alpha_s}{\pi}\right) - 3.583\left(\frac{\alpha_s}{\pi}\right)^2 - 20.215\left(\frac{\alpha_s}{\pi}\right)^3\right]\left(\pm\frac{1}{12}|g_A| + \frac{1}{36}a_8\right)$$
$$+ \left[1 - 0.333\left(\frac{\alpha_s}{\pi}\right) - 0.550\left(\frac{\alpha_s}{\pi}\right)^2 + ?\left(\frac{\alpha_s}{\pi}\right)^3\right]\frac{1}{9}\hat{a}_0$$

(not including the new results) $\qquad \alpha_s = \alpha_s(Q^2)$

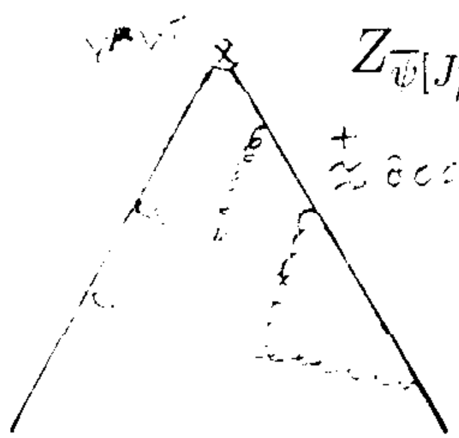
To obtain the order α_s^3 singlet coefficient:

- Calculate coefficient function C^s in 3-loops

  This is done with well established METHOD OF PROJECTORS

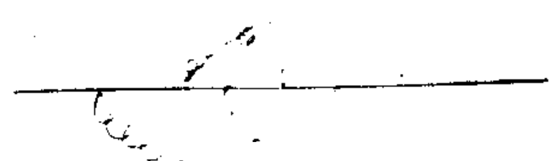
- Calculate the 4-loop contribution to the singlet anomalous dimension γ^s

  Use the auxiliary mass method (described earlier) to obtain the following renormalization factors:



$Z_{\overline{\psi}[J_\mu^5]\psi}$

+ ≈ 8000 dias.



Z_2

+ ≈ 2000 dias.

$$\gamma^s(a_s) = \frac{\partial}{\partial \ln \mu^2}\left(Z_{\overline{\psi}[J_\mu^5]\psi}/Z_2\right)$$

Results:

for $n_f = 3$ we find

$$\int_0^1 dx g_1^{p(n)} = \left[1 - \left(\frac{\alpha_s}{\pi}\right) - 3.583\left(\frac{\alpha_s}{\pi}\right)^2 - 20.215\left(\frac{\alpha_s}{\pi}\right)^3\right]\left(\pm\frac{1}{12}|g_A| + \frac{1}{36}a_8\right)$$
$$+ \left[1 - 0.333\left(\frac{\alpha_s}{\pi}\right) - 0.550\left(\frac{\alpha_s}{\pi}\right)^2 - 4.447\left(\frac{\alpha_s}{\pi}\right)^3\right]\frac{1}{9}\hat{a}_0$$

for $n_f = 4, 5, 6$:

n_f	non-singlet		singlet	
	$(\alpha_s/\pi)^2$	$(\alpha_s/\pi)^3$	$(\alpha_s/\pi)^2$	$(\alpha_s/\pi)^3$
3	-3.58333	-20.21527	-0.54959	-4.44725
4	-3.25000	-13.85026	1.08153	4.87423
5	-2.91667	-7.84019	2.97845	13.07103
6	-2.58333	-2.18506	5.27932	20.73034

Table 1. Second and third-order coefficients for the Ellis-Jaffe sum rule.