Two-Loop N=4 Susy Amplitudes and QCD

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Abstract

Two-loop four-point N=4 susy amplitudes are evaluated via cutting techniques as a testing ground for QCD. A conjecture to all loop orders is presented.

Outline

1. Status of multi-loop computations.
   Unsolved problems.
2. $N = 4$ susy as a testing ground for QCD.
3. Analytic construction of two-loop
   four-point amplitudes.
4. Conjecture for all loops.
5. Prospects for future.

Two and Higher-Loop Situation

Some of the higher loop computations that have been performed:

- $g-2$, 4 loops, Kinoshita, etc.
- $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$, $\mathcal{O}(\alpha_s^3)$ Gorishny, Kataev and Larin, etc
- Four-Loop QCD β function, Ritbergen, Vermaseren, Larin.
- Two-loop form factors, van Neerven, etc
- etc.

No two- or higher-loop amplitudes have been computed which involves more than 1 kinematic variable.

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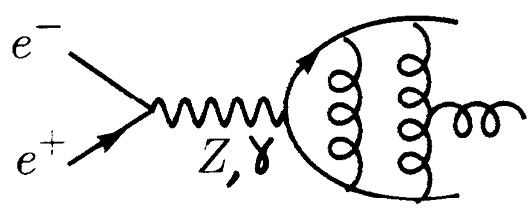
Example:



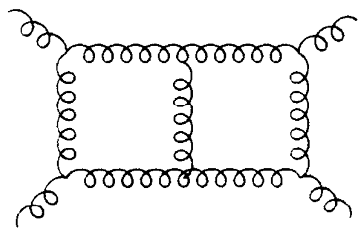
2nd example: NNLO DGLAP splitting fncs.

Important for improved measurements of α_s at LEP.

Provides motivation for investigating new calculational methods.

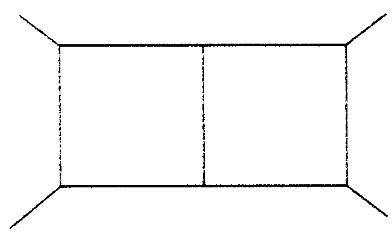
Obstacles

- Feynman diagrams a pain.



$6^6 \sim 50,000$ terms at start. Many hundreds of diagrams. Tensor integral reduction methods not so clear.

- Many integrals not known: e.g. for all massless legs



Substantial progress has however been made, especially by Ussykina and Davidychev.
Promising, but much work remains.

- Phase space integration to obtain jet cross-sections non-trivial. Infrared divergences! Must extend NLO work of Kunszt, Ellis and Soper and Giele, Glover and Kosower.

In this talk we focus on first obstacle.

Possible Approaches for Two-Loops

1. Brute force – very unsatisfying even if it can be done.
2. String-based techniques. Intriguing reorganization of amplitudes. (See L. Magnea's talk.)
3. First quantized approach being pushed by Schmidt and Schubert. Elegant evaluation of 2-loop QED β-function and Euler-Heisenberg effective action.
4. Recursive approach. (Berends & Giele; Kosower; Mahlon). Nair and Lee have developed multi-loop formalism.
5. Analytic constructions. Unitarity and cutting rules.

Here we discuss the last approach.

One-Loop Methods

Methods we wish to apply to two-loops:

- **Analytic construction based on unitarity and factorization**
- **Helicity**
- **Color decomposition**
- **Supersymmetry decompositions**

See, e.g., Z.B., D.A. Kosower and L. Dixon, Ann. Rev. Nucl. Part. Sci. 46:109 (1996) [hep-ph/9602280] for refs. and details.

Spinor Helicity

Xu, Zhang and Chang & many others

$$\varepsilon^{+}_{\mu}(k;q) = \frac{\langle q^{-} | \gamma_{\mu} | k^{-} \rangle}{\sqrt{2}\,\langle q\, k \rangle}, \qquad \varepsilon^{-}_{\mu}(k,q) = \frac{\langle q^{+} | \gamma_{\mu} | k^{+} \rangle}{\sqrt{2}\,[k\, q]}$$

All required properties of polarization vectors satisfied:

$$\varepsilon_i^2 = 0\,, \qquad k \cdot \varepsilon(k,q) = 0\,, \qquad \varepsilon^{+} \cdot \varepsilon^{-} = -1$$

Notation

$$\langle j\, l \rangle = \langle k_{j-} | k_{l+} \rangle = \sqrt{2 k_j \cdot k_l}\, e^{i\phi}$$

$$[j\, l] = \langle k_{j+} | k_{l-} \rangle = -\sqrt{2 k_j \cdot k_l}\, e^{-i\phi}$$

Adjust the reference momentum q to make terms vanish.

Equivalent to gauge transformation.

Color Decomposition

One-loop gluon amplitudes

$$A_4^{\text{oneloop}} = g^4 \sum_{\text{non-cyclic}} N_c \,\text{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]\, A_{4;1}(1,2,3,4)$$

$$+ g^4 \sum \text{Tr}[T^{a_1}T^{a_2}]\,\text{Tr}[T^{a_3}T^{a_4}]\, A_{4;3}(1,2;3,4)\,.$$

The $A_{4;j}$ are partial amplitudes.

String theory suggests:

$$A_{n;c>1} = \sum_{\text{perms}} A_{n;1}$$

Can also prove this in field theory using color ordered Feynman rules. Leading color amplitudes give everything.

Two-Loop decomposition:

$$A_4^{\text{2-loop}} = g^6 \sum N_c^2 \,\text{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}] \Big(A_{4;1,1}^{LC}(1,2,3,4)$$

$$+ \frac{1}{N_c^2} A_{4;1,1}^{SC}(1,2,3,4)\Big)$$

$$+ g^6 \sum N_c \,\text{Tr}[T^{a_1}T^{a_2}]\,\text{Tr}[T^{a_3}T^{a_4}]\, A_{4;1,3}(1,2;3,4)\,.$$

Leading color does not give everything because non-planar contributions.

Susy and QCD Amplitudes

We view susy as a tool to aid in computations.

- Supersymmetry identities (applied to QCD by Parke and Taylor)

$$\mathcal{A}_n^{\rm susy}(1_{\bar f}^-, 2^-, 3^+, \cdots, n_f^+) = \frac{\langle 2\, n\rangle}{\langle 2\, 1\rangle} \mathcal{A}_n^{\rm susy}(1^-, 2^-, 3^+, \cdots, n^+)$$

  Relates fermionic susy amplitudes to bosonic ones.

- One-loop decomposition of n-gluon amplitudes

$$\text{gluon loop:} \quad A_n^{\rm gluon} = A_n^{\rm scalar} - 4A_n^{\mathcal{N}=1} + A_n^{\mathcal{N}=4}$$

$$\text{fermion loop:} \quad A_n^{\rm fermion} = -A_n^{\rm scalar} + A_n^{\mathcal{N}=1}$$

  Each piece much easier to compute than sum.

This suggests susy will also be useful for 2 loops.

N=4 Susy Amplitudes

What is $N = 4$ super-Yang-Mills? It is ordinary Yang-Mills coupled to 4 adjoint Majorana fermions and 6 real scalars. 4 supersymmetry generators.

Amazing Properties:

1. **UV finiteness (proven by Mandelstam) when**

$$\text{Loops} < 2\,\frac{N-1}{D-4}, \qquad (Loops > 1)$$

At two-loops $N = 4$ amplitudes are UV finite for $D < 7$.

2. **Strong Susy Ward Identities (applied to QCD by Parke and Taylor).**

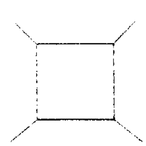
$$A_n(1^-, 2^-, \cdots, i^-, \cdots, j^-, \cdots n^-)$$
$$= \frac{\langle i\, j\rangle^4}{\langle a\, b\rangle^4} A_n(1^-, 2^-, \cdots, a^-, \cdots, b^-, \cdots n^-)$$

3. **Simple one-loop amplitudes**

$$A_{4;1}^{\text{1-loop}}(1, 2, 3, 4) = ist\, A_4^{\text{tree}}(1, 2, 3, 4)\, \mathcal{I}_4^{\text{1-loop}}(s, t)$$

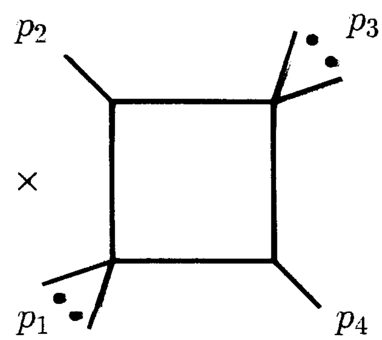
where

$$\mathcal{I}_4^{\text{1-loop}}(s, t) = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}$$



For max. helicity violation arbitrary number of external legs.

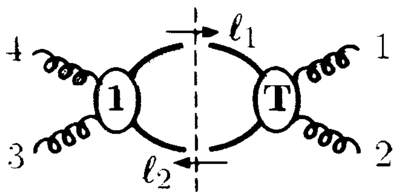
$$A_{n;1}^{\mathcal{N}=4\ \mathrm{MHV}} = -\frac{1}{2} A_n^{\mathrm{tree}} \sum \left[(p_1+p_2)^2(p_4+p_1)^2 - p_1^2 p_3^2\right] \times$$



Can we obtain similar results at two and higher loops?

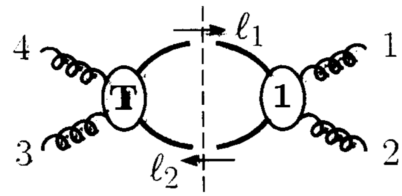
Two-Loop Cut Construction

Two-particle cuts:



(a)

$\ell_1^2 = \ell_2^2 = 0$



(b)

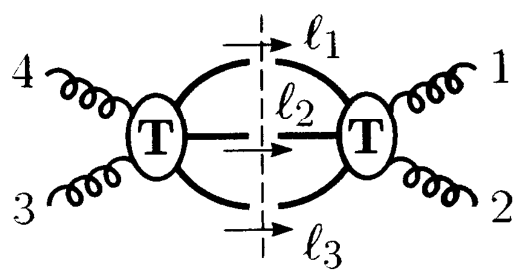
$\ell_1^2 = \ell_2^2 = 0$

$$\mathcal{A}_4^{\text{2-loop}}(1,2,3,4)\big|_{\text{cut(a)}} =$$

$$\int \sum_{P_1,P_2} \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \, \frac{i}{\ell_2^2} \, \mathcal{A}_4^{\text{1-loop}}(-\ell_2,3,4,\ell_1) \, \frac{i}{\ell_1^2} \, \mathcal{A}_4^{\text{tree}}(-\ell_1,1,2,\ell_2)\bigg|_{\ell_1^2=\ell_2^2=0}$$

This equation is valid only for those contributions which have explicit ℓ_1 and ℓ_2 propagators.

Three-particle cuts:



$\ell_1^2 = \ell_2^2 = \ell_3^2 = 0$

Reconstruct amplitude by combining all cuts into a single function with correct cuts in all channels.

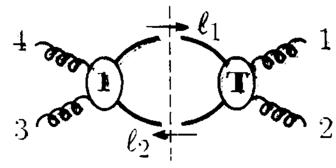
Note: By computing to all orders in the dimensional regularization parameter all terms in the amplitude must have cuts: in a massless theory every term has prefactor $(-s_{ij})^{-\epsilon}$.

Every term in a massless amplitude is detectable in the cuts.

However, for reasons of technical simplicity we want to use helicity and four-dimensional cuts. Return to this point later.

Two-Loop N=4 Amplitudes

The $s = (k_1 + k_2)^2$ channel:



$\ell_1^2 = \ell_2^2 = 0$

(a)

$$A_4^{\text{tree}}(\ell_1^+, 1^-, 2^-, \ell_2^+) = i\frac{\langle 1\,2\rangle^4}{\langle \ell_1\,1\rangle\,\langle 1\,2\rangle\,\langle 2\,\ell_2\rangle\,\langle \ell_1\,\ell_2\rangle}$$

$$A_4^{\text{tree}}(\ell_2^-, 3^+, 4^+, \ell_2^-) = i\frac{\langle \ell_1\,\ell_2\rangle^4}{\langle \ell_2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,\ell_1\rangle\,\langle \ell_2\,\ell_1\rangle}$$

The cut is

$$A_{4;1;1}^{\text{LC}}(1^-, 2^-, 3^+, 4^+)\big|_{\text{cut}}$$

$$= \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\,\frac{i}{\ell_2^2}\,A_4^{\text{1-loop}}(-\ell_2^-, 3^+, 4^+, \ell_1^-)\,\frac{i}{\ell_1^2}\,A_4^{\text{tree}}(-\ell_1^+, 1^-, 2^-, \ell_2^+)\Big|_{\ell_1^2=\ell_2^2=0}$$

$$= A_4^{\text{tree}}\Bigg[\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\,s(p+k_4)^2\,\mathcal{I}_4^{\text{1-loop}}(s, (p+k_4)^2)$$

$$\times\,\frac{\text{tr}_-[\not\ell_1 \not k_1 \not k_4 \not\ell_1 \not\ell_2 \not k_3 \not k_2 \not\ell_2]}{p^2(p-k_1)^4(p-k_1-k_2)^2(p+k_4)^4}\Bigg]\Bigg|_{\ell_1^2=\ell_2^2=0}$$

where

$$\ell_1 = p\,, \qquad \ell_2 = p - k_1 - k_2\,, \qquad A_4^{\text{tree}} = \frac{\langle 3\,4\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}$$

Sewing algebra identical to one-loop case!

Used

$$\frac{1}{\langle 4\,\ell_1\rangle} = \frac{[\ell_1\,4]}{2k_4\cdot\ell_1} = \frac{[\ell_1\,4]}{(\ell_1+k_4)^2}\,.$$

$$[\ell_1\,1]\,\langle 1\,4\rangle\,[4\,\ell_1]\,\langle \ell_1\,\ell_2\rangle\,[\ell_2\,3]\,\langle 3\,2\rangle\,[2\,\ell_2]\,\langle \ell_2\,\ell_1\rangle$$

$$= \text{tr}_-[\not\ell_1 \not k_1 \not k_4 \not\ell_1 \not\ell_2 \not k_3 \not k_2 \not\ell_2]$$

Now simplify numerator:

$$\begin{aligned}\mathrm{tr}_-[\ell_1 \not{k}_1 \not{k}_4 \ell_1 \ell_2 \not{k}_3 \not{k}_2 \ell_2] &= -4\,\mathrm{tr}_+[\not{k}_1 \not{k}_3 \not{k}_2 \not{k}_1]\,\ell_1 \cdot k_4\ \ell_1 \cdot k_1 \\ &= st(p-k_1)^2(p+k_4)^2\end{aligned}$$

where we used $\ell_1^2 = 0$, $\qquad \ell_1 \ell_2 = \ell_1 \not{k}_3 + \ell_1 \not{k}_4$

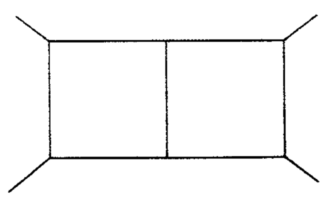
Cancels bad propagators!

Thus the cut is:

$$A_{4;1;1}^{\mathrm{LC}}(1^-, 2^-, 3^-, 4^+)\Big|_{\mathrm{cut}} = -s^2 t\, A_4^{\mathrm{tree}}(1^-, 2^-, 3^-, 4^-)\, \mathcal{I}_4^{\mathrm{P}}(s,t))\Big|_{\mathrm{cut}}$$

where scalar integral is

$$\mathcal{I}_4^{\mathrm{P}}(s,t) = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}} \times \frac{1}{p^2\,(p-k_1)^2\,(p-k_1-k_2)^2\,(p+q)^2 q^2\,(q-k_4)^2\,(q-k_3-k_4)^2}$$



Second s channel two-particle cut identical.

Two-particle t channel cuts similar except fermion and scalar loops contribute. After combining the contributions same as above, but with $s \leftrightarrow t$

Subleading Color

Must perform color decomposition on all contributions. Also non-planar contributions, but very similar to planar.

All subleading color contributions:

$$\begin{aligned}
A^{\rm SC}_{4;1,1}(1,2,3,4) = {} & 2A_4^{\rm P}(1,2;3,4) + 2A_4^{\rm P}(3,4;2,1) + 2A_4^{\rm P}(1,4;2,3) \\
& + 2A_4^{\rm P}(2,3;4,1) - 4A_4^{\rm P}(1,3;2,4) - 4A_4^{\rm P}(2,4;3,1) \\
& - 2A_4^{\rm NP}(1;2;3,4) - 2A_4^{\rm NP}(3;4;2,1) - 2A_4^{\rm NP}(1;4;2,3) \\
& - 2A_4^{\rm NP}(2;3;4,1) + 4A_4^{\rm NP}(1;3;2,4) + 4A_4^{\rm NP}(2;4;3,1)\,,
\end{aligned}$$

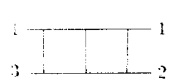
$$\begin{aligned}
A_{4;1,3}(1;2;3,4) = {} & 6A_4^{\rm P}(1,2;3,4) + 6A_4^{\rm P}(1,2;4,3) - 4A_4^{\rm NP}(1;2;3,4) \\
& - 4A_4^{\rm NP}(3;4;2,1) + 2A_4^{\rm NP}(1;4;2,3) + 2A_4^{\rm NP}(2;3;4,1) \\
& + 2A_4^{\rm NP}(1;3;2,4) + 2A_4^{\rm NP}(2;4;3,1)\,.
\end{aligned}$$

Primitive Amplitudes:

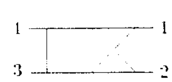
$$A_4^{\rm P}(1,2;3,4) \equiv -s_{12}^2 s_{23}\, A_4^{\rm tree}(1,2,3,4)\, \mathcal{I}_4^{\rm P}(s_{12}, s_{23})\,,$$

$$A_4^{\rm NP}(1;2;3,4) \equiv -s_{12}^2 s_{23}\, A_4^{\rm tree}(1,2,3,4)\, \mathcal{I}_4^{\rm NP}(s_{12}, s_{23})$$

$$s_{ij} \equiv (k_i + k_j)^2$$



$\mathcal{I}_4^{\rm P}$



$\mathcal{I}_4^{\rm NP}$

Validity of Cut Construction

We used $D = 4$ cuts instead of $D = 4 - 2\epsilon$ cuts. Is this valid?

Potential error terms:

$$\int \frac{d^4 p}{(2\pi)^4} \frac{d^{-2\epsilon}\mu_p}{(2\pi)^{-2\epsilon}} \frac{d^4 q}{(2\pi)^4} \frac{d^{-2\epsilon}\mu_q}{(2\pi)^{-2\epsilon}} \frac{f(p, q, k_i) \times \{\mu_p^2, \mu_q^2, \mu_p \cdot \mu_q, \cdots\}}{(p^2 - \mu_p^2)(q^2 - \mu_q^2)\cdots}$$

where μ_p and μ_q are the ϵ dimensional parts of loop momenta.

IR or UV divergence in first loop can interfere with $\mathcal{O}(\epsilon)$ in second loop leaving finite result.

Observation:

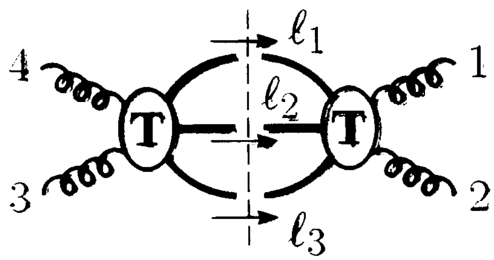
$$A_{4;1}^{\mathcal{N}=4,\ 1\text{-loop}}(1,2,3,4) = i\, st\, A_4^{\text{tree}}(1,2,3,4)\, \mathcal{I}_4^{1\text{-loop}}(s,t)$$

exact to all orders in ϵ.

One-loop algebra recycles to all orders in ϵ.

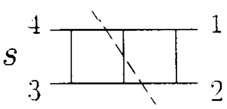
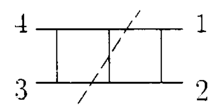
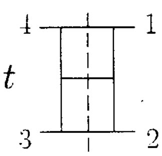
Two-particle cuts are exact to all orders in ϵ.

Three-Particle Cuts



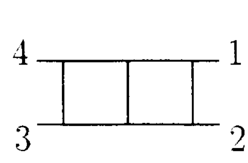
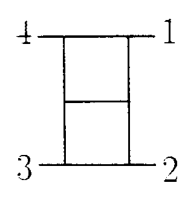
$\ell_1^2 = \ell_2^2 = \ell_3^2 = 0$

There are 16 intermediate helicity and particle configurations each given by product of tree amplitudes. $N = 4$ susy ensures that various terms combine neatly.

$-st\, A_4^{\rm tree} \Big\{ s$  $+ s$  $+ t$  $\Big\}$

Combining all cuts into a single function gives

$$A_{4;1;1}^{\rm LC}(1^-, 2^-, 3^+, 4^+)$$
$$= -st\, A_4^{\rm tree}(1^-, 2^-, 3^+, 4^+) \Big(s\, \mathcal{I}_4^{\rm P}(s,t) + t\, \mathcal{I}_4^{\rm P}(t,s) \Big)$$

$-st\, A_4^{\rm tree} \Big\{ s$  $+ t$  $\Big\}$

Consistency check: **Becomes UV divergent in $D = 7$, in agreement with superspace arguments.**

Only potential errors are in functions which have three-particle cuts but no two-particle cuts.

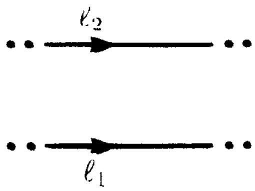
Mandelstam's $N = 4$ power counting requirements rule out the possibility of such error functions

Our two-loop cut construction is exact

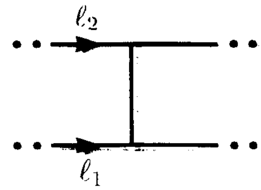
Note:

- Analytic expressions do not as yet exist for these scalar integrals.
- For QCD $\mathcal{O}(\epsilon)$ pieces are important.

All loop conjecture for leading color



$$\longrightarrow i\,(\ell_1 + \ell_2)^2 \times$$



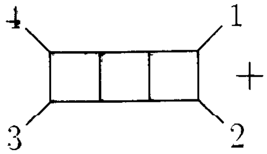
Pattern is to add extra line with given factor. No triangle or bubble sub-diagrams allowed.

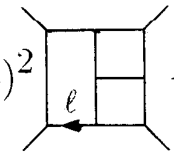
Have verified pattern consistent with two-particle cuts to all loop orders and with $D = 4$ three-particle cuts up to five loops.

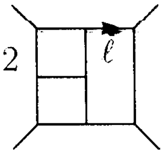
For subleading color, similar, except there are more diagrams with no two particle cuts, so conjecture less firm.

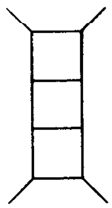
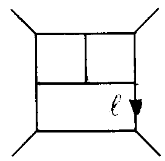
Higher Loops Conjectures

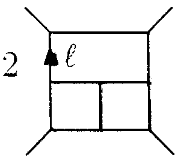
Apply same cut construction to three-loops:

$-ist\,A_4^{\rm tree}\Big\{\, s^2$


$+\, s(\ell+k_2)^2$


$+\, s(\ell+k_4)^2$


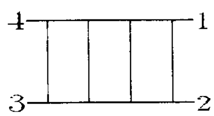
$+\, t^2$

$+\, t(\ell+k_1)^2$


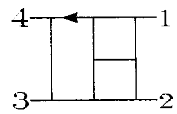
$+\, t(\ell+k_3)^2$


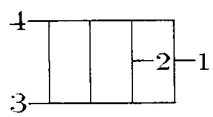
$\Big\}$

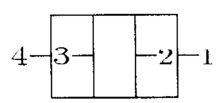
Have verified 2 and 3 particle cuts.

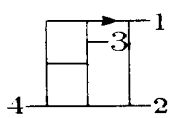
The cut construction works just as well for subleading color

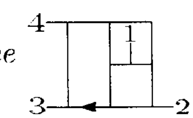
$-is^3t\,A_4^{tree}$


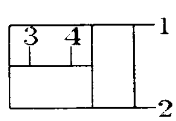
$-is^2t(\ell+k_1)^2\,A_4^{tree}$


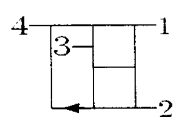
$-is^3t\,A_4^{tree}$


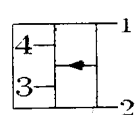
$-is^3t\,A_4^{tree}$


$-is^2t(\ell+k_3)^2\,A_4^{tree}$


$-is^2t(\ell+k_2)^2\,A_4^{tree}$


$-is^3t\,A_4^{tree}$


$-is^2t(\ell+k_2)^2\,A_4^{tree}$


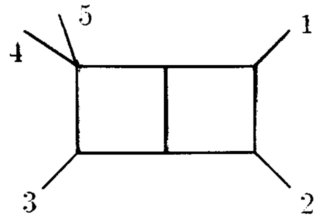
$-is^2t(\ell+k_1+k_2)^2\,A_4^{tree}$


Each diagram associated with its own color factors. Easy to **work out.**

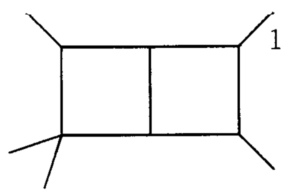
Gives conjecture for all terms in three-loop amplitude.

5-point 2-loop Conjecture

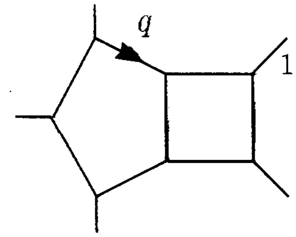
$$A_{5;1}^{2-\mathrm{loop}} = -\frac{1}{2} A_5^{\mathrm{tree}} \sum_{\mathrm{cyclic}} \Bigg\{ \; s_{12}^2 s_{23}$$



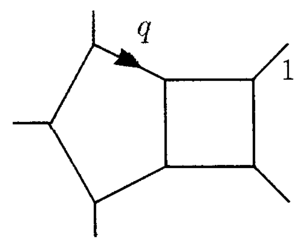
$$+ \; s_{12}^2 s_{15}$$



$$+ \; s_{12} s_{34} s_{45} \, (q - k_1)^2$$



$$+ \; s_{12} \, (q - k_1)^2 \, \mathrm{tr}_5[(\not{k}_1 + \not{k}_2) \not{k}_3 \not{q} \not{k}_5]$$



$$\Bigg\}$$

Not yet proven.

Summary

1. Experiments require two-loop computations:

   $Z \to 3$ jets.

2. $N = 4$ susy amplitudes are useful testing ground for new techniques.

3. Cutting techniques are useful at two-loops.

4. Explicit construction of N=4 two-loop amplitude in terms of scalar integral functions.

5. N=4 conjecture to all loop orders.

Cutting techniques are a promising method for obtaining 2-loop QCD amplitudes.