TWO-LOOP GLUON AMPLITUDES FROM STRING THEORY

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1. A sketch of the method at one loop.

2. Multiloop bosonic string technology.

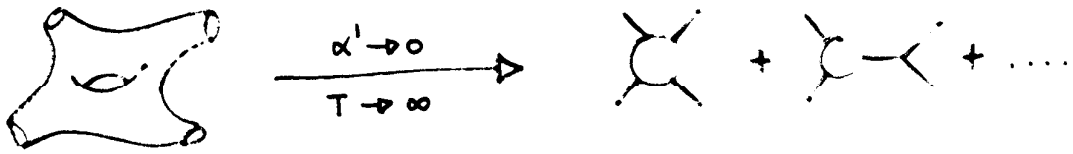
3. A laboratory: scalar amplitudes.

4. Two-loop vacuum diagrams in Yang-Mills theory

5. Outlook.

QCD scattering amplitudes from string theory (Mangano, Parke, Xu; Bern, Kosower, Dixon, Dunbar)

⊕ IDEA :



Massive string modes decouple as $M_{Pl.} \to \infty$.

⊕ THE "WISDOM" OF STRING THEORY :

Consider the colour-ordered, 1 loop, M gluon amplitude derived from the bosonic string

$$A^{(1)}(1,\dots,M) = C_1 \, n_1^M \int [dm]_1^M \prod_{i<j} e^{2\alpha' p_i \cdot p_j \, \dot{G}(\nu_i - \nu_j)}$$

$$\exp\left[\sum_{[ij]} \varepsilon_i \cdot p_j \, \dot{G}(\nu_i - \nu_j) + \frac{1}{2} \varepsilon_i \cdot \varepsilon_j \, \ddot{G}(\nu_i - \nu_j) \right]_{\text{lin.}}$$

↦ A closed expression for the M-point one-loop amplitude exists.

↦ Color decomposition has already been performed,

(Chan-Paton :

$\Rightarrow$ $N \, Tr(1 \cdots M) + Tr(1 \cdots K) \cdot Tr(K+1 \cdots M)$

subamplitudes are gauge invariant.

↦ Loop momentum integral has already been performed $\Rightarrow$ helicity methods available.

↦ Lorentz algebra is done. Feynman (Schwinger) parameter ϕ^3 type integrals are left.

$\mp$ IN PRACTICE :

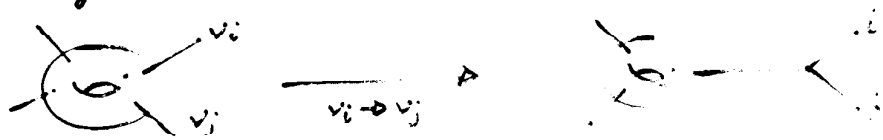
as $\alpha' \to 0$, isolate integration regions that generate explicit powers of α' compensating the ones present in the normalization.

a) "Pinching" of two gluons : $\nu_i \to \nu_j$

Then $G(\nu_i - \nu_j) \sim \ln(\nu_i - \nu_j)$

ν integrals give $\int_0 d\nu \; \nu^{2\alpha' k_i \cdot k_j - 1} \sim \dfrac{1}{2\alpha' k_i \cdot k_j}$

$\Rightarrow$ "pinching" extracts a propagator.



b) "Squeezing" the loop : $\tau \longrightarrow 0$

Then $G(\nu_i - \nu_j) \to (\nu_i - \nu_j) - \dfrac{(\nu_i - \nu_j)^2}{\tau}$

$\Rightarrow \exp[2\alpha' k_i \cdot k_j \, G(\nu_{ij})] \equiv \exp[2\alpha' k_i \cdot k_j \, (\hat{\nu}_{ij} - \hat{\nu}_{ij}^2)\tau]$

$$\Rightarrow \int_0^\infty d\tau \; \tau^{p - d/2} \, e^{-2\alpha' K(\hat{\nu})\tau} = \frac{\Gamma(p + 1 - d/2)}{[2\alpha' K(\hat{\nu})]^{p+1-d/2}}$$

The remaining ν integrals are then Feynman parameter integrals for ϕ^3-type diagrams.

<u>NOTE</u> :

$\oplus$ The results can be encoded in one-loop "Feynman" rules.

$\mp$ Dimensional regularization is easily implemented.

$\div$ complete consistency of the underlying string theory is not essential.

MULTILOOP BOSONIC STRING TECHNOLOGY.

(A. Vecchia et al.)

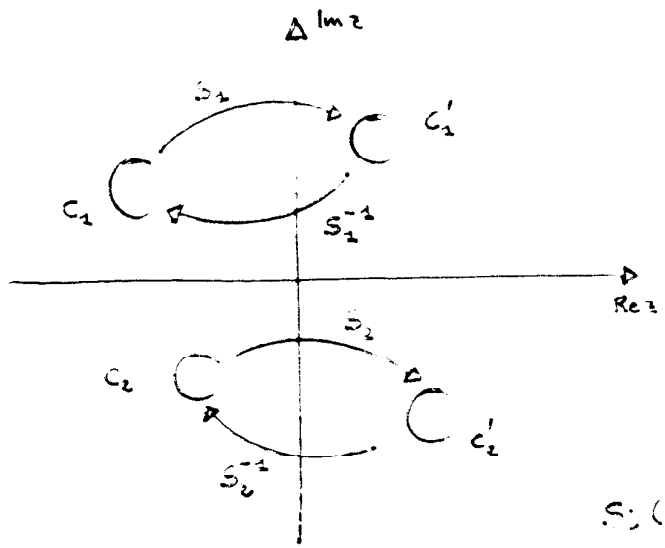
String theory is a free-field theory on the world sheet.

⇒ All correlators, including gluon amplitudes, are CALCULABLE at genus h, and can be expressed in terms of the world-sheet Green's functions.

For the bosonic string

$$\mathcal{G}_h(z_i, z_j) = \underset{\substack{\uparrow \\ \text{SHORT DISTANCE}}}{\ln(z_i - z_j)} + \underset{\substack{\uparrow \\ \text{GLOBAL}}}{R_h(z_i, z_j)}$$

$R_h(z_i, z_j)$ can be explicitly constructed $\forall h$, using the SCHOTTKY REPRESENTATION of the Riemann surface Σ_h



$$S_i(z) = \frac{a_i z + b_i}{c_i z + d_i}$$

$[\, a_i d_i - b_i c_i = 1 \,]$

OR

$$\frac{S_i(z) - \eta_i}{S_i(z) - \xi_i} = k_i \, \frac{z - \eta_i}{z - \xi_i}$$

$$S_i(C_i) = C_i' .$$

The Riemann surface is then $\Sigma_h = (\mathbb{C} \cup \infty) / \mathcal{G}_h$

where $\mathcal{G}_h$ is the Schottky group at genus h, generated by the h projective transformations $S_i(z)$.

⊕ $\{\eta_i, \xi_i\}$ are FIXED POINTS of the transformations S_i

⊕ $\{k_i\}$, the MULTIPLIERS are proportional to the radii of the circles C_i

⇒ FIELD THEORY LIMIT : $k_i \to 0$

⊕ The MODULI of Σ_h are $3h$, $(\{k_i, \eta_i, \xi_i\}) - 3$ for one overall projective transformation of the sphere.

The full h-loop bosonic Green's function is then

$$\mathcal{G}_h(z_i, z_j) = \ln E_h(z_i, z_j) - \frac{1}{2}\int_{z_i}^{z_j} \omega_\mu \left([2\pi \operatorname{Im}\tau]^{-1}\right)^{\mu\nu} \int_{z_i}^{z_j} \omega_\nu$$

where, e.g.,

$$E_h(z_i, z_j) = (z_i - z_j) \prod_\alpha \frac{T_\alpha(z_i) - z_j}{T_\alpha(z_i) - z_i} \cdot \frac{T_\alpha(z_j) - z_i}{T_\alpha(z_j) - z_j}$$

$$T_\alpha(z) \in \mathcal{G}_h \quad , \quad T_\alpha = S_i^a \circ S_j^b \circ \ldots\ldots$$

CUMBERSOME, BUT:

⇒ In the FIELD THEORY LIMIT $k_i \to 0$ ONLY A HANDFUL of T_α contribute. The relevant terms are easily generated using MATHEMATICA, MAPLE, ...

In terms of $\mathcal{G}_h$, the M-GLUON, h-LOOP BOSONIC STRING AMPLITUDE reads

$$A^{(h)}(1, \ldots, M) = C_h\, n_G^M \int [dm]_h^M \exp\left[\frac{1}{2}\sum_i p_i \cdot \varepsilon_i \frac{V_i''(0)}{(V_i'(0))^2}\right]$$

$$\prod_{i<j}\left[\frac{\exp \mathcal{G}_h(z_i, z_j)}{\sqrt{V_i'(0)\, V_j'(0)}}\right]^{p_i \cdot p_j} \cdot \exp\left[\sum_{i \neq j}\left(\varepsilon_i \cdot p_j\, \partial_{z_i} \mathcal{G}_h(z_i, z_j) + \frac{1}{2}\varepsilon_i \cdot \varepsilon_j\, \partial_{z_i}\partial_{z_j} \mathcal{G}_h(z_i, z_j)\right)\right]_{\text{LIN.}}$$

<u>NOTE:</u>

⊕ New ingredient: $V_i(z)$, projective transformations defining local coordinates around the punctures z_i

$$V_i^{-1}(z_i) = 0$$

⊕ The amplitude becomes INDEPENDENT of $V_i(z)$ when the MASS-SHELL conditions $p_i^2 = 0$; $p_i \cdot \varepsilon_i = 0$ are enforced. $A^{(h)}$ is thus a FORMAL OFF-SHELL CONTINUATION of the [illegible]

EXTENDING TO HIGHER LOOPS

I] Background field method.

At one loop string amplitudes correspond to S-matrix elements calculated in the BFM. This will remain true at all loops for external legs.

⇨ g-loop β function is encoded in the 2-point function

II] Off-shell continuation.

The off-shell extension depends on the choice of the projective transformations $\xi(z)$.

⇨ The natural geometrical choice at one loop has a natural generalization to all loops.

III] Moduli space.

Only SELECTED CORNERS of moduli space contribute to field theory amplitudes ($k_i \to 0$, $z_i \to z_j$ ...

⇨ The integration region remains simple.

IV] Schwinger parameters.

String theory gives SCHWINGER PARAMETRIZED amplitude

$$k = e^{-t/\alpha'} \quad ; \quad \hat{\nu}_i = \frac{t_1 + t_2 + \dots + t_i}{t} \quad ; \quad t = \sum_i t_i$$

⇨ NOT tied to dimensional regularization (tadpoles, vacuum bubbles...)

A LABORATORY TO UNDERSTAND
TWO LOOPS : SCALAR DIAGRAMS.

IDEA : Compute TACHYON amplitudes in bosonic string and "continue" to positive m^2

- At tree level and one loop done "consistently" in old dual model days (Scherk 1971)
- Ideal testing ground :
  - field theory result easy to calculate
  - Only leading order in k_i is needed
  - Concentrate on correspondence between individual diagrams and points in moduli space, plus integration region.

FIELD THORY LAGRANGIAN :

$$\mathcal{L} = \mathrm{Tr}\left[\partial_\mu \Phi \, \partial^\mu \Phi + m^2 \Phi^2 - \frac{\lambda}{3!} \Phi^3\right]$$

($d = 6-2\varepsilon$, $\Phi = \phi^a \lambda_a$ scalar in adjoint of $U(N)$)

STRING-THEORETIC M-POINT h-LOOP AMPLITUDE :

$$A_M^h(p_1, \dots, p_M) = N^h \, \mathrm{Tr}(\lambda^{a_1} \dots \lambda^{a_M}) \, C_h \left[2 g_s (2\alpha')^{(d-2)/4}\right]^{\lambda_h}$$

$$\cdot \int [dm]_h^M \prod_{i<j} \left[\frac{\exp(G_h(z_i, z_j))}{\sqrt{V_i'(0) V_j'(0)}}\right]^{2\alpha' p_i \cdot p_j}$$

(tachyon amplitude, planar diagrams)

CONTINUATION TO POSITIVE m^2 :

- move intercept $1 + \alpha' m^2 = 0 \Rightarrow 1 + \alpha' m^2 = 0$
- change tachyon poles in measure

$$\frac{1}{x^2} \to \frac{1}{x^{2+\alpha}} = \frac{1}{x} e^{-\alpha \ln x} = \frac{1}{x} e^{m^2 \alpha' \ln x}$$

(as in Scherk, in field theory limit).

TWO-LOOP INTEGRATION REGION:

THE TWO-ANNULUS

MODULI : $k_1, k_2, \xi_1, \xi_2, \eta_1, \eta_2, z_i$

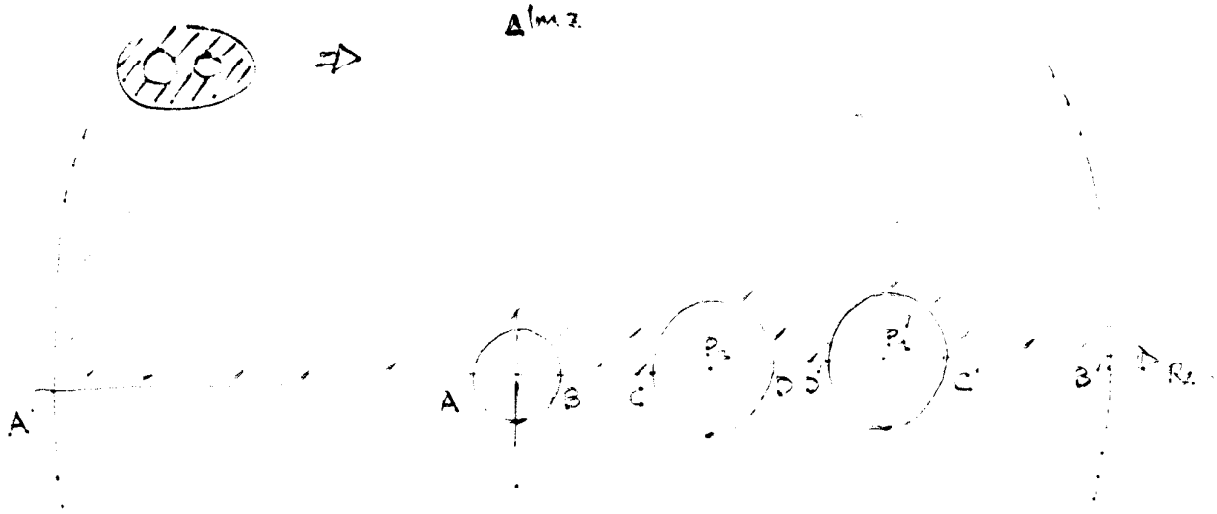
OVERALL PROJECTIVE TRANSFORMATION :

$$\eta_2 = 0 \quad ; \quad \xi_2 = \infty \quad ; \quad \xi_1 = 1 .$$

SCHOTTKY GENERATORS :

$$S_1(z) = \frac{\eta_1(z-1) - k_1(z-1)}{z-1-k_1(z-1)} \quad ; \quad S_2(z) = k_2 z .$$

REPRESENTATION OF THE ANNULUS :



with $A \sim A'$; $B \sim B'$; $C \sim C'$; $D \sim D'$

SCHOTTKY CIRCLES :

$$\mathcal{C}_2 = \{ \sqrt{k_2}\, e^{i\theta} \} \quad ; \quad \mathcal{C}_2' = \{ \frac{1}{\sqrt{k_2}} e^{i\theta} \}$$

$$P_1 = \frac{\eta_1 - k_1}{1-k_1} \quad ; \quad P_1' = \frac{1-k_1\eta_1}{1-k_1}$$

$$R_1 = R_1' = \sqrt{k_1}\, \frac{1-\eta_1}{1-k_1}$$

INTEGRATION REGION :

From condition that circles don't overlap, plus discrete symmetries

$$0 < \sqrt{k_2} < \sqrt{k_1} < \eta_1 < 1$$

with z_i's integrated over boundaries for fixed k_1, k_2, η_1.

For planar diagrams the explicit formula is

$$A_M^{(2)}(p_1,\dots,p_M) = N^2\, \mathrm{Tr}(\lambda^{a_1}\dots\lambda^{a_M})\, \frac{1}{(4\pi)^d}\, \frac{g^{2+M}}{2^{8+3M}}\, (2\alpha')^{3-d+M}$$

$$\int \frac{dk_1}{k_1^2} \int \frac{dk_2}{k_2^2} \int \frac{d\eta_1}{(1-\eta_1)^2} \int \prod_{i=1}^{M} \frac{dz_i}{V_i'(0)}$$

$$\left[\frac{1}{4}(\ln k_1 \ln k_2 - \ln^2 S)\right]^{-d/2} \cdot \prod_{i<j} \left[\frac{\exp(G_2(z_i,z_j))}{\sqrt{V_i'(0)\,V_j'(0)}}\right]^{2\alpha' p_i\cdot p_j}$$

in the "gauge" $\eta_2 = 0,\ \xi_2 = \infty,\ \xi_1 = 1$.

The Green function to leading order in k_i is

$$G_2(z_1,z_2) = \ln(z_1 - z_2) + \frac{1}{2}\, \frac{\ln^2 T \ln k_1 + \ln^2 U \ln k_2 - 2\ln T \ln U}{\ln k_1 \ln k_2 - \ln^2 S}$$

where S, T, U are anharmonic ratios

$$S = \frac{(\eta_1-\eta_2)(\xi_1-\xi_2)}{(\xi_1-\eta_2)(\eta_1-\xi_2)} \longrightarrow \eta_1$$

$$T = \frac{(z_2-\eta_1)(z_1-\xi_1)}{(z_2-\xi_1)(z_1-\eta_1)} \longrightarrow \frac{(z_2-\eta_1)(z_1-1)}{(z_2-1)(z_1-\eta_1)}$$

$$U = \frac{(z_2-\eta_2)(z_1-\xi_2)}{(z_2-\xi_2)(z_1-\eta_2)} \longrightarrow \frac{z_2}{z_1}$$

The string coupling has been replaced by the field theory coupling after computing the 3-point vertex at tree level:

$$g = 16\, g_s\, (2\alpha')^{(d-6)/4}$$

and for the projective transformations V_i we propose the choice

$$[V_i'(0)]^{-1} = \left|\frac{1}{z_i - \rho_a} - \frac{1}{z_i - \rho_b}\right|$$

with $\rho_{a,b}$ the nearest fixed points to z_i, to the left and right.

(Generalizes $V_i'(0) = z_i$ at one loop).

A SIMPLE EXAMPLE :

TWO-LOOP VACUUM BUBBLES

DIAGRAMS :

Simplest case : no external particles , no Green function
we only need to integrate the measure.

LEADING REGIONS : $k_1, k_2 \to 0$ (always) ;

$\eta_1 \to 0$, $\Rightarrow$ 1) ; $\eta_1 \to 1$, $\Rightarrow$ 1-2) .

suggested by sewing procedure , integration region and measure. $\eta_1 \sim$ "distance between the loops".

PROPER TIMES :

$\oplus$ $\eta_1 \to 1$: $k_1 = e^{-t_1/\alpha'}$; $k_2 = e^{-t_2/\alpha'}$; $1-\eta_1 = e^{-t_3/\alpha'}$

$\oplus$ $\eta_1 \to 0$: $q_1 \equiv \dfrac{k_1}{\eta_1} = e^{-t_1/\alpha'}$; $q_2 \equiv \dfrac{k_2}{\eta_1} = e^{-t_2/\alpha'}$;

$$q_3 \equiv \eta_1 = e^{-t_3/\alpha'}$$

Changing variables to proper times in the master formula without Green function , and regulating tachyon poles as indicated ($1/x^2 \to 1/x \cdot e^{m^2\alpha' \ln x}$) we find

$$= \frac{N^3}{(4\pi)^d} \frac{g^2}{2^5} \int_0^\infty dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \; e^{-m^2(t_1+t_2+t_3)} \cdot t_1 t_2 \ldots$$

$$= \frac{N^3}{(4\pi)^d} \frac{g^2}{2^5} \int_0^\infty dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \; e^{-m^2(t_1+t_2+t_3)} \cdot$$

$$\cdot (t_1 t_2 + t_1 t_3 + t_2 t_3)^{-d/2}$$

in agreement with field theory , after loop integrations , with color and symmetry factors included.

TWO-LOOP VACUUM DIAGRAMS IN YANG-MILLS THEORY

DIAGRAMS :

STRATEGY :
- Expand measure and period matrix to first order in k_1, k_2
- Examine "corners" $\eta_1 \to 0$, $\eta_2 \to 1$.
- Discard tachyons, except for contact interactions.

RESULTS : Three contributions.

a) $\eta_1 \to 0$; $q_1 = \dfrac{k_1}{\eta_1} \to 0$; $q_2 = \dfrac{k_2}{\eta_2} \to 0$

(ordered)

contributes

$$A_2^0(1) = N^3 \frac{g^2 \mu^{2\varepsilon}}{(4\pi)^d} \, d(d-2) \int_0^\infty dt_3 \int_{t_3}^\infty dt_2 \int_{t_2}^\infty dt_1 \, \frac{t_1 + t_2 + 2t_3}{(t_1 t_2 + t_1 t_3 + t_2 t_3)^{d/2+1}}.$$

NOTE : Integrand not symmetric $\Rightarrow$ not a Schwinger parameter representation.

b) $\eta_1 \sim q_2 \to 0$; $q_1 \to 0$

(as in $q_1 = e^{-t_1/\alpha'}$, $q_2 = e^{-t_2/\alpha'}$, $\eta_1 = y q_2$ with $y > 1$ finite)

Part of this contribution symmetrizes a).

$$A_2^0(1) = -N^3 \frac{g^2\mu^{2\varepsilon}}{(4\pi)^d} 2(d-2) \int_0^\infty dt_1 \int_0^{t_1} dt_2$$

$$[2t_1t_2 + t_2^2]^{-d/2} =$$

$$= N^3 \frac{g^2\mu^{2\varepsilon}}{(4\pi)^d} \Big[-2(d-2) \int_0^\infty dt_1 \int_0^{t_1} dt_2 \, (t_1t_2)^{-d/2} +$$

$$+ d(d-2) \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \frac{t_1+t_2}{[t_1t_2+t_1t_3+t_2t_3]^{d/2}}$$

The second term symmetrizes a), the first is of the type generated in field theory by the diagram ∞.

c) $\eta_1 \to 1$ $\Rightarrow$ C—C = C

However : CONTACT INTERACTION (4-point vertex).

As was done at one loop (and is common in effective action calculations in string theory), ISOLATED tachyon poles contribute their "residue", via ζ-function regularization

$$\int_0^\infty \frac{dx}{x^2} \sim \int_0^\infty dx \sum_{n=1}^\infty n e^{-nx} \sim \sum_{n=1}^\infty 1 \sim \zeta(0) = -\frac{1}{2}$$

(Note at one loop these terms are removed via integration by parts of $\ddot{G}(v_i - v_j)$).

Here we get

$$A_2^0(3) = N^3 \frac{g^2\mu^{2\varepsilon}}{(4\pi)^d} \left(-\frac{1}{2}\right)(d-2)^2 \int_0^\infty dt_1 \int_0^{t_1} dt_2 \, (t_1t_2)^{-d/2}$$

Finally : $A_2^0(1) + A_2^0(2) + A_2^0(3) = A_{2,YM}^0$.

However correspondence is NON TRIVIAL. No obvious gauge choice splits diagrams this way.

SUMMARY

- Multiloop field theory amplitudes can be consistently and efficiently calculated from the $\alpha' \to 0$ limit of string theory.

- The computation of the two-loop two-point amplitude in YM theory ($\Rightarrow$ two-loop β function) is nearly completed.

- The organization of the amplitudes in diagrams is quite different from conventional YM calculations.

- It is possible to envisage an "automatic procedure":

  INPUT:
  - Master formula
  - h-loop Green function up to terms multilinear in k_i

  STEPS:
  - Apply color decomposition and helicity methods at the very beginning.
  - List ϕ^3 and ϕ^4 diagrams contributing to the given amplitude
  - Assign "proper time" variables to each configuration and take $\alpha' \to 0$ limit
  - Scalar integrals remain to be done.

- Of interest:
  - Two-loop "rules"
  - Match with field theory methods (world-line path integrals?)

  (Bern-Dunbar; Strassler; [illegible])