RENORMALONS AT SMALL x

MICHAEL G. SOTIROPOULOS
U. OF MICHIGAN

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1. INTRODUCTION

- SMALL-x QCD AT HERA AND BEYOND ...

THEORETICAL INTEREST: THE TWO-LARGE-SCALES PROBLEM IN pQCD

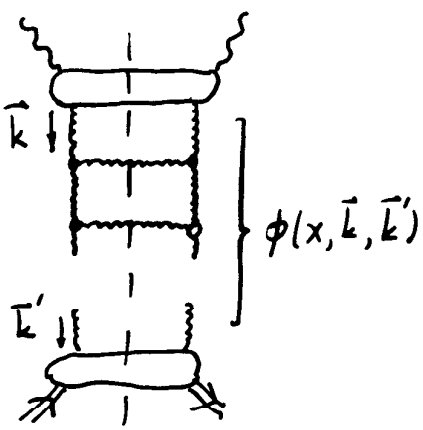
$$\Lambda \ll Q \ll \sqrt{s}\,, \qquad x := Q/\sqrt{s}$$

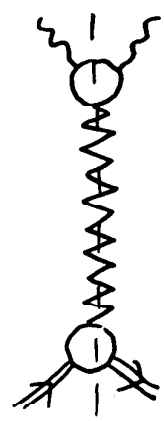
IN TERMS OF $F_2(x_B, Q^2)$, $x_B \ll 1$:
DENSE PARTONIC SYSTEMS IN THE WEAK COUPLING APPROXIMATION
(a.k.a. SEMIHARD REGIME)

FEATURES: $k_\perp$-FACTORIZATION, CONNECTION WITH REGGE LIMIT OF S-MATRIX

LLA(x) FOR SEMIHARD PROCESSES MEANS SUMMATION OF LARGE $\ln\frac{1}{x}$ AS:

$$\varphi(x, \vec{k}^2/Q^2) = \sum_{n=0} \alpha_s^n \sum_{m=0}^{n} \ln^m \frac{1}{x}\, F_{nm}\left(\frac{\vec{k}^2}{Q^2}\right) \xrightarrow{LLA(x)} \sum_n \alpha_s^n \ln^n \frac{1}{x}\, F_{nn}\left(\frac{\vec{k}^2}{Q^2}\right)$$





REGGE DESCRIPTION

$$\left(\frac{1}{x}\right)^{\alpha_{I\!P} - 1}$$

$\alpha_{I\!P}$: "POMERON INTERCEPT"

BFKL '76

$$-x \frac{\partial}{\partial x} \phi(x, \vec{k}, \vec{k}') = \int d^2k_1\, L(\vec{k}, \vec{k}_1)\, \phi(x, \vec{k}_1, \vec{k}')$$

- BFKL KERNEL IS INFRARED REGULAR AND ITS ACTION IS CONFORMALLY INVARIANT

DIAGONALIZATION: $\phi(\vec{k}^2) = (\vec{k}^2)^{-\gamma}$, $\quad \int d^2\vec{k}_1\, L(\vec{k},\vec{k}_1)\,\phi(\vec{k}_1^2) = \frac{\alpha_s N_c}{\pi}\,\chi(\gamma)\,\phi(\vec{k}^2)$

SPECTRUM: $\chi(\gamma) = 2\,\psi(1) - \psi(\gamma) - \psi(1-\gamma)$

BFKL "POMERON INTERCEPT": $\alpha_{I\!P} = 1+\lambda$, $\quad \lambda = \frac{4\alpha_s N_c}{\pi}\ln 2 \simeq 0.5$

COMPARE WITH "SOFT POMERON INTERCEPT" $\lambda = 0.08$ DONNACHIE LANDSHOFF

- PROBLEMS WITH LLA(x): 1) UNITARITY 2) IR DIFFUSION

IN LLA(x) ASSUME: $\vec{k}'^2 \simeq \vec{k}_1^2 \simeq \ldots \simeq \vec{k}^2 = Q^2$ AND FIXED $\alpha_s = \alpha_s(Q^2)$
LARGE α_s UNCERTAINTIES COME FROM DIFFUSION OF $\langle \vec{k}^2 \rangle$ TOWARDS IR
TREATMENT OF IR REGION REQUIRES RUNNING α_s (CONSISTENTLY IN NLLA(x))

ESTIMATION OF CONSTRAINTS FOR 1) AND 2) FROM ASYMPTOTIC SOLUTION MUELLER '9
hep-ph/9612251

UNITARITY CONSTRAINT: $Y \lesssim \frac{2}{\alpha_{I\!P}-1}\ln\frac{1}{\alpha_s(k_0^2)}$

DIFFUSION CONSTRAINT: $Y \lesssim \frac{\pi}{14 N_c\,\zeta(3)\,\beta_0^2}\,\frac{1}{\alpha_s^3(k_0^2)}$; $\beta_0 = \frac{1}{4\pi}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$

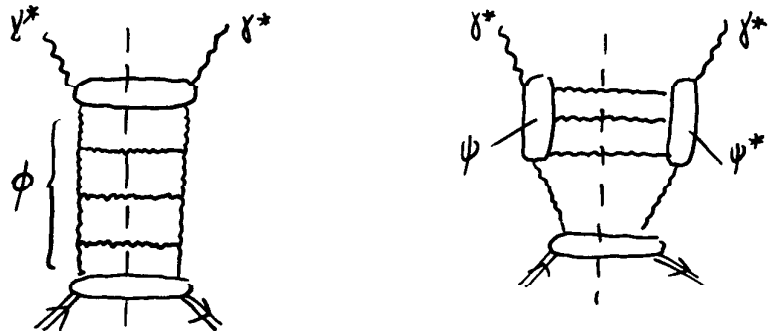
WHERE $k_0 \sim 1/R$, R: (SMALL) SIZE OF SCATTERING BOUND STATE
FOR $\alpha_s(k_0) \ll 1$ UNITARITY CONSTRAINT REACHED BEFORE DIFFUSION CONSTRAINT
BUT "... IN MOST PRACTICAL CIRCUMSTANCES DIFFUSION CONSTRAINTS WILL BE REACHED BEFORE UNITARITY CONSTRAINTS."

2. THE DIPOLE APPROACH TO SMALL-X PHYSICS

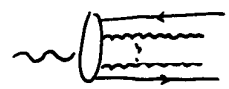
A. MUELLER '94

N. NIKOLAEV
B. ZAKHAROV '94

BFKL (t-CHANNEL) $\longleftrightarrow$ DIPOLE (s-CHANNEL)

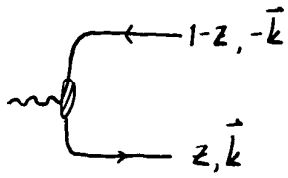


IN DIPOLE APPROACH WE CONSTRUCT THE EVOLUTION FOR $\psi^*\psi$ WHERE

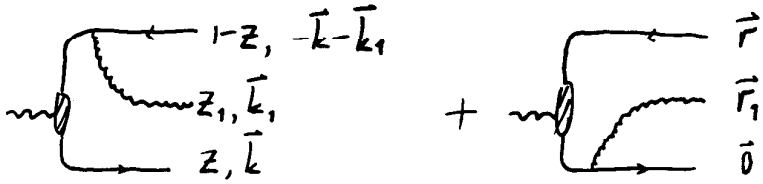
$\psi^{(n)} =$  $\}$ n soft gluons

i.e. THE n-SOFT GLUON COMPONENT OF THE (ONIUM) WAVE FUNCTION.

THE LOWEST ORDER $\mathcal{O}(g_s^0)$ WAVEFUNCTION IS THAT OF A COLOR DIPOLE

$\psi^{(0)}(z,\vec{k}) =$ 

THE $\mathcal{O}(g_s)$ WAVEFUNCTION DESCRIBES THE EMISSION OF ONE SOFT GLUON

$\psi^{(1)}(z, z_1, \vec{k}, \vec{k}_1) =$ 

$z_1 \ll z$

$\hat{r}_1 = \vec{r} - \vec{r}_1$

EXPLICITLY, IN LIGHT-CONE GAUGE

$$\psi^{(1)a}(z,z_1;\vec{k},\vec{k}_1) = -2g_s T^a \left[\psi^{(0)}(z,\vec{k}) - \psi^{(0)}(z,\vec{k}+\vec{k}_1)\right] \frac{\vec{k}_1\cdot\vec{\epsilon}}{\vec{k}_1^{\,2}}$$

FOURIER XFORM IN IMPACT PARAM. SPACE

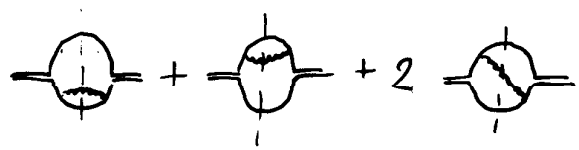
$$\psi^{(1)a}(z,z_1;\vec{r},\vec{r}_1) = -ig\,\frac{T^a}{\pi}\,\psi^{(0)}(z,\vec{r})\left(\frac{\vec{r}_1}{\vec{r}_1^{\,2}} - \frac{\vec{\hat{r}}_1}{\vec{\hat{r}}_1^{\,2}}\right)\cdot\vec{\epsilon}$$

THEREFORE, THE PROBABILITY FOR ONE GLUON (ONE DIPOLE, $N_c\to\infty$) EMISSION

$$\Phi^{(1)}(z,\vec{r}) \equiv \int\frac{d^2\vec{r}_1}{2\pi}\int^{z}\frac{dz_1}{2z_1}\sum\psi^{(1)*}\psi^{(1)} = \int^{z}\frac{dz_1}{z_1}\int d^2\vec{r}_1\,\frac{\alpha_s C_F}{\pi^2}\,\frac{\vec{r}^{\,2}}{\vec{r}_1^{\,2}(\vec{r}-\vec{r}_1)^2}\,\Phi^{(0)}(z,\vec{r})$$

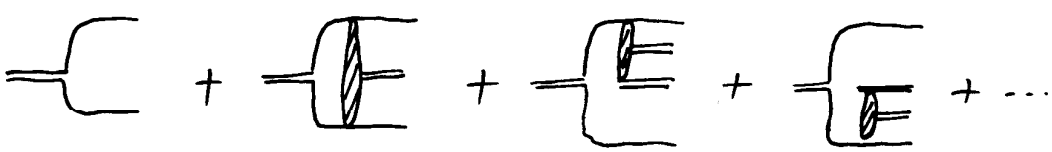
ANATOMY OF THE KERNEL:

$$\frac{\vec{r}^{\,2}}{\vec{r}_1^{\,2}(\vec{r}-\vec{r}_1)^2} = \frac{1}{\vec{r}_1^{\,2}} + \frac{1}{(\vec{r}-\vec{r}_1)^2} - 2\,\frac{\vec{r}_1\cdot(\vec{r}-\vec{r}_1)}{\vec{r}^{\,2}(\vec{r}-\vec{r}_1)^2}$$



NOTE: THE DIPOLE KERNEL IS MANIFESTLY IR REGULAR AND SCALE INVARIANT

IN THE LARGE N_c LIMIT, MULTIPLE DIPOLE EMISSION FACTORIZES SIMILARLY



• SPECTRAL ANALYSIS:

EVOLUTION EQUATION:

$$\frac{\partial}{\partial Y}\tilde{n}(Y,r,p;b) = \int_0^\infty dr' \int_0^b db' \, \tilde{K}(r,r';b') \, \tilde{n}(Y,r',p;b-b')$$

BOUNDARY CONDITION: $\tilde{n}(Y=0,r,p;b) = r\,\delta(r-p)\,\delta(b)$

INTRODUCE LAPLACE (MELLIN) XFORM

$$\tilde{n}_{\omega,\gamma}(b) = \int_0^\infty dY\, e^{-\omega Y} \int_0^\infty \frac{dr^2}{r^2}\left(\frac{r^2}{p^2}\right)^{-\gamma} \tilde{n}(Y,r,p;b)$$

AND OBTAIN EQUATION FOR SPECTRAL AMPLITUDES $\tilde{n}_{\omega,\gamma}(b)$

$$\omega\,\tilde{n}_{\omega,\gamma}(b) = 2\delta(b) + \frac{N_c}{\pi}\int_0^b db' \, \chi(\gamma-b'\beta_0, b') \left(\frac{Q^2p^2}{4}\right)^{b'\beta_0} \tilde{n}_{\omega,\gamma-b'\beta_0}(b-b')$$

INTEGRAL EQUATION OF VOLTERRA TYPE. NO EIGENFUNCTIONS FOR BOUNDED KERNEL. SOLUTION BY ITERATION.

$$\tilde{n}_{\omega,\gamma}(b) = \sum_{k=0}^{\infty} \tilde{n}^{(k)}_{\omega,\gamma}(b)$$

WITH $\tilde{n}^{(0)}_{\omega,\gamma} = \frac{2}{\omega}\delta(b)$, $\quad \tilde{n}^{(1)}_{\omega,\gamma}(b) = \frac{2}{\omega}\left(\frac{N_c}{\pi\omega}\right)\left(\frac{Q^2p^2}{4}\right)^{b\beta_0}\chi(\gamma-b\beta_0,b)$

AND AFTER k-ITERATIONS ... →

$$\tilde{n}^{(k)}_{\omega,\gamma}(b) = \frac{2}{\omega}\left(\frac{N_c}{\pi\omega}\right)^k \left(\frac{Q^2p^2}{4}\right)^{b\beta_0} \int_0^b db_{k-1}\, \chi(\gamma - b_{k-1}\beta_0, b_{k-1})$$

$k \geq 2$

$$\times \int_0^{b-b_{k-1}} db_{k-2}\, \chi(\gamma - (b_{k-1}+b_{k-2})\beta_0, b_{k-2}) \cdots$$

$$\times \int_0^{b-b_{k-1}\cdots-b_2} db_1\, \chi(\gamma - (b_{k-1}\cdots+b_1)\beta_0, b_1)\, \chi(\gamma - b\beta_0, b-b_{k-1}\cdots-b_1).$$

$k-1$ NESTED INTEGRATIONS. SINCE $\chi(\gamma, b)$ IS KNOWN WE CAN STUDY THE SINGULARITY STRUCTURE OF $\tilde{n}^{(k)}_{\omega,\gamma}(b)$ ... CONSIDER $0 < \gamma < 1$

- <u>NET RESULT</u>: SPECTRAL AMPLITUDE IS ANALYTIC IN b IN THE REGION $0 < b < \gamma/\beta_0$ WITH LEADING IR SINGULARITY A BRANCH CUT AT $b = \gamma/\beta_0$

RECALL, IN THE ASYMPTOTIC LIMIT $Y \to \infty$, FROM SADDLE POINT APPROX $\gamma = \frac{1}{2} + iV$. SUCH $Re(\gamma) = \frac{1}{2}$ IMPLIES PRESENCE OF $O\left(\frac{\Lambda}{Q}\right)$ POWER CORRECTION

THESE HAVE ALSO BEEN DETECTED STARTING FROM CONVENTIONAL BFKL + RUNNING COUPLING $\frac{\alpha_s(\vec{k}'^2)\ \alpha_s((\vec{k}-\vec{k}')^2)}{\alpha_s(\vec{k}^2)}$ BY LEVIN '95 NPB 453 (1995) 303

- FROM ITERATIVE SOLUTION OBTAIN THE b-EXPANSION FOR $\tilde{n}^{(k)}_{\omega,\gamma}(b)$ $\longrightarrow$

$$\tilde{n}^{(k)}_{\omega,\gamma}(b) = \frac{2}{\omega}\left(\frac{N_c}{\pi\omega}\right)^k \Bigg\{ \frac{b^{k-1}}{(k-1)!}\,\chi(\gamma)^k +$$

$$\frac{b^k}{(k-1)!}\left[\beta_0 \ln\left(\frac{Q^2p^2}{4}\right)\chi(\gamma)^k + \beta_0\,\chi^{(1)}(\gamma)\;\chi(\gamma)^{k-1}\right] +$$

$$-\frac{b^k}{2}\left[\frac{1}{(k-2)!} + \frac{2}{(k-1)!}\right]\beta_0\,\chi(\gamma)^{k-1}\chi'(\gamma) + O(b^{k+1})\Bigg\}$$

SUMMATION OF LEADING POWERS OF b (I.E. RETAIN ONLY $O(b^{k-1})$ IN ABOVE) IS REQUIRED IN THE REGION

$$b \ll 1 \;, \quad \frac{N_c\,\chi(\gamma)}{\pi\omega}\, b \sim 1$$

THIS RESUMMATION OF $O(N_c\,\chi(\gamma)\,b/\pi\omega)$ TERMS IS WELL DEFINED IF $O(b^k) \ll O(b^{k-1})$. FOR THE $\ln Q^2$ TERM THIS TRANSLATES TO

$$b\beta_0 \ln\frac{Q^2p^2}{4} \ll 1 \Rightarrow b\left(\frac{1}{\alpha_s(Q^2)} - \frac{1}{\alpha_s(1/p^2)}\right) \ll 1$$

WE RECOVER THE CONFORMAL LIMIT BY LEADING-b RESUMMATION

$$n_{\omega,\gamma} = \sum_{k=0}^{\infty} \frac{2}{\omega}\left[1 + \frac{N_c\,\alpha_s\,\chi(\gamma)}{\pi\omega} + \left(\frac{N_c\,\alpha_s\,\chi(\gamma)}{\pi\omega}\right)^2 + \dots\right] = \frac{2}{\omega - N_c\,\alpha_s\,\chi(\gamma)/\pi}$$

INVERSE LAPLACE AND SADDLE POINT:

$$n(Y,r,p) = \frac{1}{2}\,\frac{r}{p}\,\frac{e^{-(\alpha_{I\!P}-1)Y}}{\sqrt{\pi D Y}}\,\exp\left(-\frac{\ln^2(r/p)}{4DY}\right)$$

$$D = \frac{1}{\pi}\,\frac{7}{2}\,\zeta(3)\,\alpha_s N_c$$

BUT...

NOT ALL PHYSICS IS IN THE UNIVERSAL DIPOLE DENSITY,
IT IS ALSO IN THE IMPACT FACTORS.

CONSIDER, FOR EXAMPLE SMALL-x D.I.S.
THE DIPOLE DENSITY MUST BE FOLDED WITH THE $(\text{WAVE FUNCTION})^2 = \phi^{(0)}$
FOR $\gamma^* \to q\bar{q}$.

$$\Phi_T^{(0)}(z,r) = \frac{2N_c e^2 Z_f^2}{(2\pi)^2} \left\{ [z^2 + (1-z)^2]\lambda^2 K_1(\lambda r)^2 + m_f^2 K_0(\lambda r)^2 \right\}$$

$$\phi_L^{(0)}(z,r) = \frac{8N_c e^2 Z_f^2}{(2\pi)^2} z^2(1-z)^2 Q^2 K_0(\lambda r)^2$$

THE SCALE IS SET BY $\lambda^2 = z(1-z)Q^2 + m_f^2$

THE STRUCTURE FUNCTION AND ITS BOREL IMAGE ARE RELATED AS USUAL

$$F_{T,L}(x,Q^2) = \int_0^\infty db\, \tilde{F}(x,Q^2;b)\, e^{-b/\alpha_s(Q^2)}$$

THE BOREL IMAGE IS GIVEN BY

$$\tilde{F}_{T,L}(x,Q^2;b) = \frac{Q^2}{4\pi\alpha_{em}} \int_x^1 dz \int_0^\infty dr^2\, \tilde{\Phi}_{T,L}(z,r;b)\, \sigma_0(r, m_N)$$

DIPOLE NUCLEON INTERACTION

THE DIPOLE CASCADE EVOLVES AS:

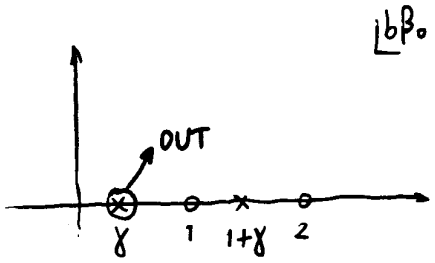
$$z\frac{\partial}{\partial z}\tilde{\Phi}_{T,L}(z,r;b) = z\frac{\partial}{\partial z}\phi^{(0)}_{T,L}(z,r)\,\delta(b) + \int_0^\infty dr' \int_0^b db'\, \tilde{K}(r,r';b)\, \tilde{\Phi}_{T,L}(z,r';b-b')$$

IR RENORMALONS ARISE FROM $r' \gg r$. RECALL ASYMPTOTIC EXPANSION

$$K_\nu(x) \xrightarrow{x\to\infty} \sqrt{\frac{\pi}{2x}}\, e^{-x}\left[1 + O\left(\tfrac{1}{x}\right)\right]$$

IN THE RANGE OF INTEREST $r' \to \infty$, $\Phi^{(0)}_{T,L}(z,r') \sim e^{-2\lambda r'}$
IT HAS A SHAPE, UNLIKE THE BASIS FUNCTIONS $(r'^2)^{\gamma}$

THIS WILL CHANGE THE IR RENORMALON STRUCTURE OF THE DIPOLE CASCADE EMANATING FROM γ^*



THE LEADING IR RENORMALON IS AT $b\beta_0 = 1$
HENCE $\frac{1}{Q^2}$ POWER CORRECTIONS (CONSISTENT WITH WILSON OPE EXPECTATION)

THIS HAS BEEN OBSERVED NUMERICALLY FROM THE ITERATIVE CONSTRUCTION OF THE SOLUTION.

5. CONCLUSIONS AND OUTLOOK

- WE HAVE IMPLEMENTED THE RUNNING COUPLING IN THE SMALL-x RESUMMATION OF THE PERTURBATIVE SERIES, TO STUDY EFFECTS FROM IR DIFFUSION

- OUR APPROACH IS BASED ON TWO (SO FAR DISJOINT) INPUTS:
  1) THE DIPOLE FORMALISM FOR SMALL-x
  2) RENORMALON ANALYSIS OF THE PERTURBATIVE SERIES

- WE HAVE CONSTRUCTED AN EVOLUTION EQ. FOR THE DIPOLE DENSITY IN BOREL SPACE WHICH ADMITS SYSTEMATIC EXPANSION AROUND THE CONFORMAL LIMIT $b=0$.

- FOR SMALL-x DIS STRUCTURE FUNCTIONS WE SEE $1/Q^2$ POWER CORRECTIONS

IN PROGRESS...

- CONSTRUCT NUMERICALLY THE SOLUTION FOR γ^* DIPOLE CASCADE
- EXAMINE HOW THESE IR EFFECTS AFFECT THE BFKL POMERON INTERCEPT (TOWARDS RECONCILING "HARD" - "SOFT" POMERON)
- COMPARE WITH FULL ANSWER i.e. NLLA(x) FADIN, LIPATOV '96
- INTERPRET OUR RESULTS IN TERMS OF A GENERALIZED HIGH ENERGY OPE BALITSKY '96

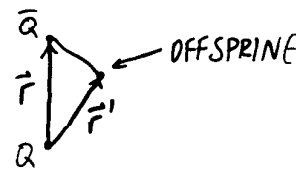
IT IS USEFUL TO INTRODUCE THE UNIVERSAL (PROJECTILE AND TARGET INDEP.) DIPOLE DENSITY $n(Y, r, p)$, SUCH THAT

MUELLER '94
PATEL
NPB425 (1994) 47

$$\Phi(z, r^2) = \Phi^{(0)}(z, r^2) \cdot \int \frac{d^2\vec{p}}{2\pi p}\, n(\overset{=\ln z}{Y}, r, p)$$

THE DIPOLE DENSITY OBEYS THE EVOLUTION EQUATION

$$n(Y, r, p) = \underbrace{r\delta(r-p)}_{n_0} + \int^{Y} dy \int_0^{\infty} dr'\, K(r, r')\, n(y, r', p)$$



WITH KERNEL

$$K(r, r') = -\frac{\alpha_s N_c}{2\pi^2}\,\delta(r-r') \int d^2\vec{s}\, \frac{\vec{r}^{\,2}}{\vec{s}^{\,2}(\vec{r}-\vec{s})^2} + \frac{\alpha_s N_c}{\pi^2} \int_0^{\infty} d|\vec{r}-\vec{r}'|\, J(\vec{r}, \vec{r}')\, \frac{\vec{r}^{\,2}}{\vec{r}'^{\,2}(\vec{r}-\vec{r}')^2}$$

VIRTUAL CORRECTION — REAL EMISSION

EQUIVALENT FORM (AMENABLE TO MONTE CARLO-SIMULATION, OEDIPUS (SALAM '96) C. SCHMIDT

$$n(Y, r, p) = \exp\left(-\frac{2\alpha_s N_c}{\pi} \ln\frac{r}{r_{min}}\, Y\right) n_0 +$$

$$+ \frac{2\alpha_s N_c}{\pi} \int^{Y} dy\, \exp\left(-\frac{2\alpha_s N_c}{\pi} \ln\frac{r}{r_{min}} (Y-y)\right) \int d^2\vec{r}'\, \frac{\vec{r}^{\,2}}{\vec{r}'^{\,2}(\vec{r}-\vec{r}')^2}\, n(y, r', p)$$

THE SPECTRUM OF THE DIPOLE KERNEL IS THE SAME AS THE BFKL KERNEL

$$\int_0^{\infty} dr'\, K(r, r')\, (r'^2)^{\gamma} = \chi(\gamma)(r^2)^{\gamma} \quad \rightarrow \text{SAME LLA(x) PHENOMENOLOGY}$$

THE EQUIVALENCE BETWEEN DIPOLE AND BFKL KERNELS IS SHOWN BY ANALYZING DIAGRAMS IN T.O.P.T.

CHEN-MUELLER '95
NPB451 (1995) 579

3. THE DIPOLE KERNEL WITH RUNNING COUPLING

ANDERSON
ROSS
M.G.S. '96
PLB 380 (1996) 127

PROBLEM: INCLUDE SYSTEMATICALLY THE RUNNING COUPLING IN LLA(x) AND STUDY THE ENSUING IR SINGULARITY STRUCTURE

APPROACH:

- START WITH DIPOLE FORMULATION OF LLA(x) IN IMPACT PARAMETER SPACE BECAUSE ITS CLEAR PROBABILISTIC INTERPRETATION (CASCADE BRANCHING)
- PRESCRIPTION FOR RUNNING THE COUPLING:

  IN MOM. SPACE: $\alpha_s \longrightarrow \alpha_s(\vec{k}^2)$ DOKSHITZER'S PLENARY TALK

  (MOTIVATION FROM SOFT GLUON RESUMMATION IN TIMELIKE CASCADES '94
  IT WORKS FOR GLUON REGGEIZATION KORCHEMSKY, STERMAN M.G.S.
- MAIN PROBLEM:

  HOW TO TRANSLATE $\alpha_s(\vec{k}^2)$ CONSISTENTLY IN IMPACT PARAMETER SPACE
- TECHNIQUE

  USE THE BOREL XFORM IN THE INTERMEDIATE STEPS AS A GUIDE TO SELECT THE APPROPRIATE FUNCTION OF DIPOLE SIZES THAT SETS THE α_s SCALE.

WF IN MOM SPACE $g_s(\vec{k}^2)$ $\xrightarrow{\text{BOREL}}$ BOREL IMAGE
$\gamma^* \to q\bar{q}$

$\downarrow$ FOURIER AND SQUARE $\otimes$

PROBABILITY WITH RUNNING COUPLING $\xleftarrow{\text{BOREL}^{-1}}$ BOREL IMAGE OF PROBABILITY IN IMP. PARAMETER SPACE

$\downarrow$

DIPOLE EV. KERNEL IN BOREL SPACE

$$\Phi(z, r^2; \alpha_s) = \int_0^\infty db\, \tilde{\Phi}(z, r^2; b)\, e^{-b/\alpha_s(Q^2)}$$

PROBABILITY FOR DIPOLE SPLITTING WITH RUNNING α_s:

$$\Phi^{(1)}(z,r^2) = \frac{C_F}{\pi^3} \ln z \int_0^\infty d\tau\, d\tau'\, J_1(\tau)\, J_1(\tau') \int_0^1 \frac{d\omega}{\omega^{1/2}(1-\omega)^{1/2}} \times$$

$$\int d^2\vec{r}_1 \frac{1}{\vec{r}_1^{\,2}(\vec{r}-\vec{r}_1)^2} \left\{ \alpha_s\left(\frac{\lambda^2}{R^2}\right)\vec{r}^{\,2} + \left[\alpha_s\left(\frac{\lambda^2}{r_1^2}\right) - \alpha_s\left(\frac{\lambda^2}{R^2}\right)\right](\vec{r}-\vec{r}_1)^2 + \right.$$

$$\left. + \left[\alpha_s\left(\frac{\lambda^2}{(\vec{r}-\vec{r}_1)^2}\right) - \alpha_s\left(\frac{\lambda^2}{R^2}\right)\right]\vec{r}_1^{\,2} \right\} \Phi^{(0)}(z,r^2)$$

WHERE: $R^2 = (\vec{r}_1^{\,2})^{\omega}(\vec{r}-\vec{r}_1)^{2(1-\omega)}$

$\lambda^2 = (\tau^2)^{\omega}(\tau'^2)^{(1-\omega)}$

- CHECK THAT FOR α_s: FIXED IT REPRODUCES THE KNOWN EXPRESSION
- THE BOREL IMAGE OF $\Phi^{(1)}$, AFTER PERFORMING ALL INTEGRATIONS, IS:

$$\tilde{\Phi}^{(1)}(z,r^2;b) = -\frac{2C_F}{\pi} \ln z \, \frac{\Gamma(-b\beta_0)}{\Gamma(1+b\beta_0)} \left(\frac{Q^2 r^2}{4}\right)^{b\beta_0} \Phi^{(0)}(z,r^2) \qquad (*)$$

IDENTIFY "USUAL" IR RENORMALON POLES AT $b\beta_0 = 1,2,3,\ldots$

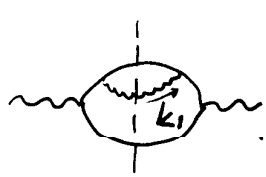
FOR D.Y. KORCHEMSKY-STERMAN '95 NPB437(1995)415

THE SINGULARITY AT $b\beta_0 = 0$ IS OF UV ORIGIN, REGULATED BY CUTOFF r_{min}

<u>CONSISTENCY CHECK:</u> $\Phi^{(1)}(z,r^2)$ CAN ALSO BE EXPRESSED AS:

$$\Phi^{(1)}(z,r^2) = 4C_F \ln z \; \Phi^{(0)}(z,r^2) \int \frac{d^2\vec{k}_1}{\vec{k}_1^{\,2}} \alpha_s(\vec{k}_1^{\,2}) \left(2 - e^{-i\vec{k}_1\cdot\vec{r}} - e^{i\vec{k}_1\cdot\vec{r}}\right)$$

WHICH YIELDS THE SAME $\tilde{\Phi}^{(1)}(z,r^2;b)$ AS IN (*)



END RESULT:

DOKSHITZER
URALTSEV '96

EVOLUTION EQUATION FOR THE DIPOLE DENSITY IN BOREL SPACE

$$\frac{\partial}{\partial Y}\tilde{n}(Y,r,p;b) = \int_0^\infty dr' \int_0^b db' \, \tilde{K}(r,r';b')\, \tilde{n}(Y,r',p;b-b')$$

WITH BOUNDARY CONDITION $\tilde{n}(Y=0,r,p;b) = r\,\delta(r-p)\,\delta(b)$

THE DIPOLE KERNEL IN BOREL SPACE IS:

$$\tilde{K}(r,r';b) = \frac{N_c}{\pi}\left(\frac{Q^2 r^2}{4}\right)^{b\beta_0} \frac{\Gamma(-b\beta_0)}{\Gamma(1+b\beta_0)}\,\delta(r-r') + \qquad \text{VIRTUAL}$$

$$2\frac{N_c}{\pi^2}\frac{1}{r'}\left(\frac{Q^2 r'^2}{4}\right)^{b\beta_0} \int_0^1 \frac{d\omega}{\omega^{1/2}(1-\omega)^{1/2}} \frac{\Gamma(1-\omega b\beta_0)\,\Gamma(1-(1-\omega)b\beta_0)}{\Gamma(1+\omega b\beta_0)\,\Gamma(1+(1-\omega)b\beta_0)} \times$$

$$\left\{ \left(\frac{r_>^2}{r'^2}\right)^{b\beta_0-1} F\left(1-b\beta_0;\, 1-b\beta_0;\, 1;\, \frac{r_<^2}{r_>^2}\right) + \right.$$

$$\left(\frac{r^2-r'^2}{r_>^2}\right)\left(\frac{r_>^2}{r'^2}\right)^{\omega b\beta_0} F\left(1-\omega b\beta_0;\, 1-\omega b\beta_0;\, 1;\, \frac{r_<^2}{r_>^2}\right) + \qquad \text{REAL}$$

$$\left. -\left(\frac{r_>^2}{r'^2}\right)^{\omega b\beta_0} F\left(-\omega b\beta_0;\, -\omega b\beta_0;\, 1;\, \frac{r_<^2}{r_>^2}\right) \right\}$$

WHERE $r_{><} = \begin{matrix}\max\\ \min\end{matrix}\{r,r'\}$

THIS FORM OF $\tilde{K}$ IS AMENABLE TO RENORMALON ANALYSIS.
IR RENORMALONS, FOR INSTANCE, ARE EXPECTED TO ARISE FROM THE REGION $r'>r$, WHERE $F \equiv {}_2F_1$ HAS ANALYTIC POWER EXPANSION.

4. BOREL SINGULARITIES OF THE DIPOLE CASCADE

- IS THE KERNEL $\tilde{K}(r,r';b)$ SINGULAR AT $b \to 0$ AND FOR FIXED PARENT r ?
  THE $b \to 0$ SINGULARITY IN THE REAL PART IS OF UV ORIGIN.
  IT COMES FROM ${}_2F_1(1-b\beta_0, 1-b\beta_0; 1; J)$ WITH $J = r_<^2/r_>^2 \to 1$
  USE LINEAR XFORM:

$$ {}_2F_1(1-b\beta_0, 1-b\beta_0; 1; J) = (1-J)^{-1+2b\beta_0} \, {}_2F_1(b\beta_0, b\beta_0; 1; J) $$

IN THE LIMIT $b \to 0$ THE KERNEL BECOMES

$$ \tilde{K}(r,r';b) = \frac{N_c}{\pi}\left(\frac{Q^2 r^2}{4}\right)^{b\beta_0} \left\{ \frac{1}{-b\beta_0}\,\delta(r-r') + \frac{2}{r'}(1-J)^{-1+2b\beta_0} + \right. $$
$$ \left. + \text{ UV regular} \right\} $$

USING: $$ (1-J)^{-1+2b\beta_0} = \frac{1}{2b\beta_0}\,\delta(1-J) + \left[\frac{1}{1-J}\right]_+ + \cdots $$

THE $b \to 0$ SINGULARITY CANCELS BETWEEN REAL AND VIRTUAL CONTRIBUTIONS !

- IF THERE EXISTS REGION OF ANALYTICITY IN THE BOREL PLANE IT WILL CONTAIN THE ORIGIN.

CONSIDER THE ACTION OF $\tilde{K}$ ON TEST FUNCTION $(r^2)^\gamma$

$$\int_0^\infty dr' \, \tilde{K}(r,r';b)\,(r'^2)^\gamma = \frac{N_c}{\pi}\,\chi(\gamma,b)\left(\frac{Q^2 r^2}{4}\right)^{b\beta_0}(r^2)^\gamma$$

NON-CONFORMALLY INVARIANT ACTION

FROM EXPLICIT CALCULATION THE "SPECTRAL" FUNCTION $\chi(\gamma,b)$ IS:

$$\chi(\gamma,b) = \underbrace{\frac{\Gamma(-b\beta_0)}{\Gamma(1+b\beta_0)}}_{\text{VIRTUAL}} + \frac{\Gamma(-\gamma-b\beta_0)}{\Gamma(1+\gamma+b\beta_0)}\,\frac{1}{\pi}\int_0^1 \frac{d\omega}{\omega^{1/2}(1-\omega)^{1/2}}\,\frac{\Gamma(1-\omega b\beta_0)\,\Gamma(1-(1-\omega)b\beta_0)}{\Gamma(1+\omega b\beta_0)\,\Gamma(1+(1-\omega)b\beta_0)}\times$$

$$\times\left[\underbrace{\frac{\Gamma(1+\gamma)}{\Gamma(-\gamma)}\,\frac{\Gamma(b\beta_0)}{\Gamma(1-b\beta_0)}}_{\text{DIRECT}} - \underbrace{2\,\frac{\Gamma(1+\omega b\beta_0)}{\Gamma(1-\omega b\beta_0)}\,\frac{\Gamma(1+\gamma+(1-\omega)b\beta_0)}{\Gamma(1-\gamma-(1-\omega)b\beta_0)}}_{\text{INTERFERENCE}}\right]$$

OBSERVE γ-DEPENDENT IR RENORMALONS AT $b\beta_0 = n-\gamma$, $n=1,2,\ldots$ COMING FROM THE $r'>r$ INTEGRATION REGION.

SINCE $b=0$ IS REGULAR POINT, $\chi(\gamma,b)$ ADMITS b-EXPANSION

$$\chi(\gamma,b) = -\frac{1}{b\beta_0} + \frac{1}{b\beta_0} + \underbrace{\chi(\gamma)}_{\text{BFKL}} + b\beta_0\,\chi^{(1)}(\gamma) + \underbrace{O(b^2\beta_0^2)}_{\text{KNOWN}}$$

WHERE:

$$\chi^{(1)}(\gamma) = -\frac{1}{\gamma}\,\chi(\gamma) - 2\psi(1)\,\chi(\gamma) + \frac{1}{2}\,\chi(\gamma)^2 + \frac{1}{2}\,\chi'(\gamma)$$