Relativistic QM + units

Schrödinger eq: $i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = H |\psi, t\rangle$

$H = \frac{\vec{p}^{\,2}}{2m}$, and $\vec{p} = \frac{\hbar}{i} \nabla \Rightarrow i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) = \frac{-\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t)$

in position space

Relativistic Hamiltonian: $H = \sqrt{m^2c^4 + p^2c^2} \Rightarrow \sqrt{m^2 + p^2}$

with $c = 1$

$\Rightarrow i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) = \sqrt{m^2 - \hbar^2 \nabla^2}\ \psi(\vec{x}, t)$

Now set $\hbar = 1 \Rightarrow i \frac{\partial}{\partial t} \psi(\vec{x}, t) = \sqrt{m^2 - \nabla^2}\ \psi(\vec{x}, t)$

Everything now measured in units of mass

$[m] = +1$ $\quad [t] = -1$ $\qquad$ Will use from now on

$[\ell] = -1$ $\quad [\frac{\partial}{\partial t}] = +1$

$[\nabla] = +1$

also treats time, space differently

If nothing else, eq is ugly $\Rightarrow$ infinite # spatial derivative

Think of as differential operators applied to ψ on both sides $\Rightarrow$ square before applying

$$\Rightarrow \left(i \frac{\partial}{\partial t}\right)^2 \psi(\vec{x}, t) = \left\{\sqrt{m^2 - \nabla^2}\right\}^2 \psi(\vec{x}, t)$$

$$\Rightarrow \boxed{-\frac{\partial^2}{\partial t^2} \psi(\vec{x}, t) = \left\{m^2 - \nabla^2\right\} \psi(\vec{x}, t)}$$

Klein-Gordon eq

Review of Special Relativity

We'll see that the K.G. eq gives the same description of physics in all inertial frames ⇒ First, a review of SR

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x^1, x^2, x^3)$$

$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (x^0, -x^1, -x^2, -x^3)$$

~~(Note: different from Srednicki here!)~~

Transform between contravariant (upper) and covariant (lower) indices using metric tensor

$$x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu = g_{\mu\nu} x^\nu$$ ~~(Einstein summation convention)~~

Can work out that

$$g_{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Similarly, $x^\mu = g^{\mu\nu} x_\nu$ with $g^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

Work out $g_{\mu\nu} g^{\nu\rho} = \delta_\mu{}^\rho = \begin{pmatrix} +1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$ identity matrix

Note $x_\mu x^\mu = t^2 - \vec{x}^2$ ⇒ the relativistic interval

Perform a Lorentz transformation on x

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$$

You know, for example, what a boost looks like ($\hat{z}$ direction)

$$\bar{x}^0 = \gamma x^0 + \gamma \beta x^3 \qquad \beta = \frac{v}{c}, \; \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\bar{x}^3 = \gamma \beta x^0 + \gamma x^3$$

Relativistic invariance: interval must be the same after L.T.

$$\bar{x}^\mu \bar{x}_\mu = \Lambda^\mu{}_\nu x^\nu g_{\mu\rho} x^\rho = \Lambda^\mu{}_\nu x^\nu g_{\mu\rho} \Lambda^\rho{}_\sigma x^\sigma$$

$$= \underbrace{\Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\sigma}_{\text{must be } g_{\nu\sigma} \text{ to get } x^\mu x_\mu} x^\nu x^\sigma$$

$\Rightarrow$ L.T. must satisfy $\boxed{\Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\sigma = g_{\nu\sigma}}$

Let's see now what happens to our K.G. eq under L.T.

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right\} \psi(\vec{x}, t) = 0$$

Set $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{\partial t}, \nabla \right)$

Then, following our rules for raising/lowering indices

$$\partial^\mu = g^{\mu\nu} \partial_\nu = \left(\frac{\partial}{\partial t}, -\nabla \right)$$

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \partial^2 \Rightarrow \text{can write K.G. eq as}$$

$$\{\partial^2 + m^2\}\Psi(x) = 0$$

How does ∂^μ transform under L.T ?

Use chain-rule: $\bar{\partial}^\mu = \frac{\partial}{\partial \bar{x}_\mu} = \frac{\partial x_\nu}{\partial \bar{x}_\mu} \frac{\partial}{\partial x_\nu}$

We know $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow$ multiply both sides by $g_{\mu\rho}\Lambda^\rho{}_\sigma$ and use L.T. condition

$$\Rightarrow x_\sigma = \bar{x}_\rho \Lambda^\rho{}_\sigma$$

Taking a derivative shows $\frac{\partial x_\sigma}{\partial \bar{x}_\rho} = \Lambda^\rho{}_\sigma$

$\Rightarrow \bar{\partial}^\mu = \Lambda^\mu{}_\nu \partial^\nu \Rightarrow$ exactly what blind index manipulation would indicate

Similarly, $\bar{\partial}_\mu \bar{\partial}^\mu = \Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\sigma \partial^\nu \partial^\sigma = \partial_\mu \partial^\mu$

Make a L.T. from $x \to \bar{x}$; wave-function in principle goes from $\Psi(x) \to \bar{\Psi}(\bar{x})$

$\{\partial^2 + m^2\}\Psi(x) = 0$; in the new frame

$\{\bar{\partial}^2 + m^2\}\bar{\Psi}(\bar{x}) = 0$ But this becomes

$$\{\partial^2 + m^2\}\bar{\Psi}(\bar{x}) = 0$$

Since they obey identical eqs. of motion,

$$\bar{\Psi}(\bar{x}) = \Psi(x) \Rightarrow \text{same physics in different frames}$$

(Note: $x, \bar{x}$ are the same space-time point, just measured in different coordinate systems)

$\boxed{\text{Problems with the K.G. eq.}}$

Let's write down some solutions. Consider

$$\Psi(x) = e^{-ip\cdot x}, \text{ with } p\,x = p^0x^0 - p^1x^1 - p^2x^2 - p^3x^3$$

$$\partial_\mu \Psi(x) = -ip_\mu e^{ip\,x}$$

$$\partial^2 \Psi(x) = -p^2 e^{ip\,x}$$

$$\Rightarrow \{\partial^2 + m^2\}\Psi(x) = 0 \quad \text{as long as } p^2 = m^2$$

$$\Rightarrow p^0 = \pm\sqrt{\vec{p}^2 + m^2}$$

Momentum of state: $\langle\Psi|\frac{1}{i}\nabla|\Psi\rangle = \vec{p}$

Energy: $\langle\Psi| i\frac{\partial}{\partial t}|\Psi\rangle = p^0$

$\Rightarrow$ not necessarily positive

Negative energy solutions imply no ground-state

$\Rightarrow$ Keep adding more & more negative energy particles

The K-G eq doesn't describe a single particle. It describes a field which can create / destroy particles at a given space-time point. We'll see how this interpretation arises.

We'll begin by writing a Lagrangian for the K-G field. In classical mechanics, $\underset{\text{action}}{S} = \int dt \underset{\text{Lagrangian}}{L}$

Since $\phi(\vec{x},t) \equiv \phi(x)$ depends on x, we'll write a Lagrangian density
(change from ψ)

$$S = \int dt\, d^3x\, \mathcal{L} \equiv \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

Action had better be Lorentz-invariant. Under L.T.,

$d^4\bar{x} = |\det \Lambda|\, d^4x \Rightarrow$ From $\Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\sigma = g_{\nu\sigma}$, can derive $|\det \Lambda| = +1$

$\Rightarrow \mathcal{L}$ had better be a scalar.

In point-particle mechanics, eq. of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \Rightarrow \text{becomes}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)$$

$\Rightarrow$ $\boxed{\text{derive this}}$

Pretty easy to check that $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \Omega_0$

gives K.G. eq.

We know the solutions: $e^{i\vec{k}\cdot\vec{x} \pm i\omega t}$

with $\omega = \sqrt{\vec{k}^2 + m^2}$

$\Rightarrow$ not a single particle state, so we don't mind the $\pm\omega$

General solution: superposition of plane waves

$$\phi(\vec{x},t) = \underbrace{\int \frac{d^3k}{F(k)}}\left\{a(\vec{k})\,e^{i\vec{k}\cdot\vec{x} - i\omega t} + b(\vec{k})\,e^{i\vec{k}\cdot\vec{x} + i\omega t}\right\}$$

we'll use $F(k)$ to introduce the Lorentz-invariant Phase-space

$\Uparrow$ L.I: $\bar{k}^2 = k^2$

Consider $\int d^4k\, \delta(k^2 - m^2)\,\Theta(k^0)$

$\Downarrow$

L.I: $d^4\bar{k} = |\det\Lambda|\, d^4k = d^4k$

$\Theta(k^0)$ is also L.I. a proper L.T won't change the sign of k^0

$$\int d^4k\, \delta(k^2 - m^2)\,\Theta(k^0) = \int_{-\infty}^{\infty} dk^0 \int d^3k\; \delta[(k^0)^2 - \vec{k}^2 - m^2]\;\Theta(k^0)$$

Use δ function to do k^0 integral

$k^0 = \pm\sqrt{\vec{k}^2+m^2} = \pm\omega \Rightarrow$ however, only + solution survives
$\theta(k^0)$

Use $\int dx\, \delta[f(x)] = \int dx \frac{\delta(x)}{|f'(x)|}$ to write

$$\int_{-\infty}^{\infty} dk^0 \int d^3k\ \delta[(k^0)^2 - \vec{k}^2 - m^2]\,\theta(k^0) = \int \frac{d^3k}{2\omega}$$

$\Rightarrow$ if we choose $F(k) = 2\omega$, our P.S. is L.I.

$$\Rightarrow \phi(x) = \int \underbrace{\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}}_{L.I}\left\{\underbrace{a(\vec{k})}_{L.I}\, \underbrace{e^{i\vec{k}\cdot\vec{x} - i\omega t}}_{L.I} + \underbrace{b(\vec{k})}_{L.I.}\, e^{i\vec{k}\cdot\vec{x} + i\omega t}\right\}$$

Let's make our scalar field real by imposing
$\phi^*(x) = \phi(x)$

$$\phi^*(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left\{a^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x} + i\omega t} + b^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x} - i\omega t}\right\}$$

take $\vec{k} \to -\vec{k}$ in each term

$$= \int \frac{d^3k}{(2\pi)^3 2\omega}\left\{a^*(-\vec{k})\, e^{i\vec{k}\cdot\vec{x} + i\omega t} + b^*(-\vec{k})\, e^{i\vec{k}\cdot\vec{x} - i\omega t}\right\}$$

Equate integrands $\Rightarrow$ $b(\vec{k}) = a^*(-\vec{k})$ for $\phi(x) = \phi^*(x)$
for ϕ, ϕ^*

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left\{ a(\vec{k})\, e^{i\vec{k}\cdot\vec{x} - i\omega t} + \underbrace{a^*(-\vec{k})\, e^{i\vec{k}\cdot\vec{x}}}_{\text{take } \vec{k} \to -\vec{k}}{}^{+i\omega t} \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \left\{ a(\vec{k})\, e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x} + i\omega t} \right\}$$

Introduce $k^\mu = (\omega, \vec{k})$, and we have

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left\{ a(\vec{k})\, e^{-ik\cdot x} + a^*(\vec{k})\, e^{ik\cdot x} \right\}$$

This is the mode expansion of the real scalar field, & every term is explicitly L.I.

Quantization of the real scalar field

We'll impose the canonical commutation relations on the field & its conjugate momentum $\Rightarrow \vec{x}$ just a label now; it has been demoted

Momentum density: $\Pi(x) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(x)$

Hamiltonian density: $\mathcal{H} = \Pi \dot{\phi} - \mathcal{L}$

$$= \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 - \Omega_0$$

The Hamiltonian is $H = \int d^3x\, \mathcal{H}(x) \Rightarrow$ work this out using the mode expansion

$$\Pi = \int \frac{d^3k}{(2\pi)^3 2\omega} \left\{ -i\omega a(\vec{k}) e^{-ik\cdot x} + i\omega a^*(\vec{k}) e^{ik\cdot x} \right\}$$

$$\nabla\phi = \int \frac{d^3k}{(2\pi)^3 2\omega} \left\{ i\vec{k}\, a(\vec{k}) e^{-ik\cdot x} - i\vec{k}\, a^*(\vec{k}) e^{ik\cdot x} \right\}$$

Each term in H:

(1) $\frac{1}{2}\Pi^2$: $\frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega} \int \frac{d^3k'}{(2\pi)^3 2\omega'} \int d^3x \left\{ -i\omega a(\vec{k}) e^{-ik\cdot x} + i\omega a^*(\vec{k}) e^{ik\cdot x} \right\} \left\{ -i\omega' a(\vec{k}') e^{-ik'\cdot x} + i\omega' a^*(\vec{k}') e^{+ik'\cdot x} \right\}$

Recall $\int \frac{dx}{2\pi} e^{ikx} = \delta(k) \Rightarrow \int \frac{d^3x}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} = \delta^{(3)}(\vec{k})$

$$\Rightarrow \frac{1}{8} \int \frac{d^3k}{(2\pi)^3} \left\{ -a(\vec{k}) a(-\vec{k}) e^{-2i\omega t} + a(\vec{k}) a^*(\vec{k}) + a^*(\vec{k}) a(\vec{k}) - a^*(\vec{k}) a^*(-\vec{k}) e^{2i\omega t} \right\}$$

(2) $\frac{1}{2}(\nabla\phi)^2$: $\frac{1}{8} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega^2} \left\{ \vec{k}^2 a(\vec{k}) a(-\vec{k}) e^{-2i\omega t} + \vec{k}^2 a(\vec{k}) a^*(\vec{k}) + \vec{k}^2 a^*(\vec{k}) a(\vec{k}) + \vec{k}^2 a^*(\vec{k}) a^*(-\vec{k}) e^{2i\omega t} \right\}$

(3) $\frac{1}{2}m^2\phi^2$: $\frac{1}{8} \int \frac{d^3k}{(2\pi)^3} \frac{m^2}{\omega^2} \left\{ a(\vec{k}) a(-\vec{k}) e^{-2i\omega t} + a(\vec{k}) a^*(\vec{k}) + a^*(\vec{k}) a(\vec{k}) + a^*(\vec{k}) a^*(-\vec{k}) e^{2i\omega t} \right\}$

(4) $-\int d^3x\, \Omega_0 = -\Omega_0 V$

$\uparrow$ volume of space

Sum these:

$$H = -\Omega_0 V + \frac{1}{8}\int \frac{d^3k}{(2\pi)^3}\Big\{ e^{-2i\omega t}\Big[-1 + \frac{\vec{k}^2}{\omega^2} + \frac{m^2}{\omega^2}\Big] a(\vec{k})a(-\vec{k})$$

$$+ e^{2i\omega t}\Big[-1 + \frac{\vec{k}^2}{\omega^2} + \frac{m^2}{\omega^2}\Big] a^*(\vec{k})a^*(-\vec{k})$$

$$+ \Big[1 + \frac{\vec{k}^2}{\omega^2} + \frac{m^2}{\omega^2}\Big]\Big[a(\vec{k})a^*(\vec{k}) + a^*(\vec{k})a(\vec{k})\Big]\Big\}$$

with $\omega^2 = \vec{k}^2 + m^2$, the Hamiltonian becomes

$$H = -\Omega_0 V + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega \left[a(\vec{k})a^*(\vec{k}) + a^*(\vec{k})a(\vec{k})\right]$$

Now we're ready to quantize. Impose

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0 \quad \Rightarrow \quad [x, y] = 0$$

$$[\pi(\vec{x},t), \pi(\vec{x}',t)] = 0 \quad \Rightarrow \quad [p_x, p_y] = 0$$

$$[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^{(3)}(\vec{x}-\vec{x}') \Rightarrow [x, p_x] = i\hbar$$

What do these imply for $a(\vec{k}), a^*(\vec{k})$? First solve for a, a^* in terms of ϕ, π

$$\Rightarrow \phi(\vec{x},0) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left\{ a(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a^+(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right\}$$

$$\Pi(\vec{x},0) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left\{ -i\omega a(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + i\omega a^*(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right\}$$

Multiply both expressions by $e^{-i\vec{k}\cdot\vec{x}}$, integrate over x

$$\Rightarrow \int d^3x\, \phi(\vec{x},0) e^{-i\vec{k}'\cdot\vec{x}} = \frac{1}{2\omega'} \left\{ a(\vec{k}') + a^+(-\vec{k}') \right\}$$

$$\int d^3x\, \Pi(\vec{x},0)\, e^{-i\vec{k}'\cdot\vec{x}} = \frac{1}{2} \left\{ -ia(\vec{k}) + ia^+(-\vec{k}) \right\}$$

Multiply top expression by ω', bottom by i; add them

$$a(\vec{k}) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \left\{ \omega \phi(\vec{x},0) + i\,\Pi(\vec{x},0) \right\}$$

$$a^+(\vec{k}) = \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \left\{ \omega \phi(\vec{x},0) - i\,\Pi(\vec{x},0) \right\}$$

Now work out the commutators for a, a^+

(1) $$[a(\vec{k}), a(\vec{k}')] = \int d^3x\, d^3y\, e^{-i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{y}} \left[\omega \phi(\vec{x},0) + i\,\Pi(\vec{x},0),\ \omega' \phi(\vec{y},0) + i\,\Pi(\vec{y},0) \right]$$

$$= \int d^3x\, d^3y\, e^{-i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{y}} \Big\{ i\omega \underbrace{[\phi(\vec{x},0), \Pi(\vec{y},0)]}_{i\delta^{(3)}(\vec{x}-\vec{y})} - i\omega' \underbrace{[\phi(\vec{y},0), \Pi(\vec{x},0)]}_{i\delta^{(3)}(\vec{x}-\vec{y})} \Big\}$$

$$= \underbrace{\int d^3x\, e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')} \{\omega' - \omega\} = 0$$

(2) $[a^+(\vec{k}), a^+(\vec{k}')] = 0 \Rightarrow$ just conjugate previous one

(3) $[a(\vec{k}), a^+(\vec{k}')] = \int d^3x\, d^3y\, e^{-i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{y}}$

$$[\omega\phi(\vec{x},0) + i\Pi(\vec{x},0),\ \omega'\phi(\vec{y},0) - i\Pi(\vec{y},0)]$$

$$= \int d^3x\, d^3y\, e^{-i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{y}} \{\omega\delta^{(3)}(\vec{x}-\vec{y}) + \omega'\delta^{(3)}(\vec{x}-\vec{y})\}$$

$$= (2\pi)^3 2\omega\, \delta^{(3)}(\vec{k}-\vec{k}')$$

These should look Familiar. Do you recall the raising/lowering operators for the non-relativistic S.H.O. ? They had $[a, a^+] = 1 \Rightarrow$ this is the continuum version

These a/a^+ also destroy/create quanta of energy. We will interpret them as particles. $a^+(\vec{k})$ will create a particle with momentum $(\omega, \vec{k})$.

Let's go back to look at H.

$$H = -\Omega_0 V + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega} \, \omega\left[a(\vec{k})a^+(\vec{k}) + a^+(\vec{k})a(\vec{k})\right]$$

We will normal order this; a^+ on left, a on right

Replace $a(\vec{k})a^+(\vec{k}) = a^+(\vec{k})a(\vec{k}) + (2\pi)^3 2\omega \delta^{(3)}(0)$

$$\Rightarrow \delta^{(3)}(0) = \int \frac{d^3x}{(2\pi)^3} = \frac{V}{(2\pi)^3} \rightarrow \text{space volume}$$

$$H = \underbrace{\int \frac{d^3k}{(2\pi)^3 2\omega} \, \omega \, a^+(\vec{k})a(\vec{k})}_{\substack{a^+a \text{ counts \# of} \\ \text{particles; this} \\ \text{just sums energies}}} + V\left\{\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega - \Omega_0\right\}$$

$$\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{\vec{k}^2+m^2} = \frac{1}{4\pi^2}\int_0^{\Lambda \leftarrow \text{cutoff}} dk \, \vec{k}^2 \sqrt{\vec{k}^2+m^2} \simeq \frac{\Lambda^4}{16\pi^2}$$

$\Rightarrow$ this diverges

This is the energy of the vacuum, along with Ω_0.

Taking $\langle 0|H|0\rangle = V\left\{\frac{\Lambda^4}{16\pi^2} - \Omega_0\right\}$

Unless we put in gravity, these terms have no effect $\Rightarrow$ we'll drop them

Note on Lorentz transformations

Recall the time evolution of a Heisenberg operator

$$e^{iHt}\phi(\vec{x},0)e^{-iHt} = \phi(\vec{x},t)$$

Define the 4-vector operator $\hat{P}^\mu = (H, \hat{\vec{P}}) \Rightarrow$ must have

$$e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x} = \phi(x) \quad \Rightarrow T(x) = e^{i\hat{P}\cdot x}$$

is a space-time translation operator

$$T(x)\phi(0)T(x)^{-1} = \phi(x)$$

Introduce a Lorentz transformation operator $U(\Lambda)$

$$\Rightarrow U(\Lambda)\phi(x)U(\Lambda)^{-1} = \phi(\Lambda x) \equiv \phi(\bar{x})$$

same space-time point measured in new coordinate system

Apply to mode expansion

$$U(\Lambda)\phi(x)U(\Lambda)^{-1} = \int \frac{d^3k}{(2\pi)^3 2\omega}\left\{a(\vec{k})e^{-ik\cdot(\Lambda x)} + a^+(\vec{k})e^{ik\cdot(\Lambda x)}\right\}$$

Dot products in exponent are $k_\mu \bar{x}^\mu = g_{\mu\rho}k^\rho \Lambda^\mu{}_\nu x^\nu$

Shift integration: $k^\rho = \Lambda^\rho{}_\sigma k'^\sigma \Rightarrow$ measure invariant exponent becomes $k'\cdot x$

$$\Rightarrow \int \frac{d^3k'}{(2\pi)^3 2\omega}\left\{a(\Lambda\vec{k}')e^{-ik'\cdot x} + a^+(\Lambda\vec{k}')e^{ik'\cdot x}\right\}$$

From this we derive the transformation properties of $a, a^\dagger$:

$$U(\Lambda)a(\vec{k})U(\Lambda)^{-1} = a(\Lambda\vec{k})$$
$$U(\Lambda)a^\dagger(\vec{k})U(\Lambda)^{-1} = a^\dagger(\Lambda\vec{k})$$

Now consider a state $a^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$

$$U(\Lambda)\left[a^\dagger(\vec{k})|0\rangle\right] = \underbrace{U(\Lambda)a^\dagger(\vec{k})U(\Lambda)^{-1}}_{a^\dagger(\Lambda\vec{k})}\ \underbrace{U(\Lambda)|0\rangle}_{|0\rangle}$$
$$= |\Lambda\vec{k}\rangle$$

With similar logic, consider $[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0$ and L.T. $\Rightarrow U(\Lambda)[\phi(\vec{x},t), \phi(\vec{x}',t)]U(\Lambda)^{-1} = 0$

$$= [U(\Lambda)\phi(\vec{x},t)U(\Lambda)^{-1}, U(\Lambda)\phi(\vec{x}',t)U(\Lambda)^{-1}] = 0$$

$\Rightarrow$ Any field separated by a distance obtainable from $-(\vec{x}-\vec{x}')^2$ by L.T. must vanish

$\Rightarrow [\phi(x), \phi(x')] = 0$ for $(x-x')^2 < 0$

Causality & the spin-statistics thm

We would like our theory to be causal $\Rightarrow$ measurements at spacelike separated points ($(x-x')^2<0$) shouldn't influence each other. We can build measurement functions out of products of field operators $\Rightarrow$ for example, the energy is quadratic in ϕ, Π

Let's see what causality of bi-local operators such as $\mathcal{O}(x) = \phi(x)\phi(x)$ implies

$[\mathcal{O}(x), \mathcal{O}(x')] = 0$ for $(x-x')^2 < 0$ $\Rightarrow$ measurements don't influence each other.

Our K.G describes bosons: $[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0$

leads to $[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$

$\Rightarrow$ can exchange order of two particles

If we were dealing with fermions, we'd have

$$\{a^\dagger(\vec{k}), a^\dagger(\vec{k}')\} = 0 \Rightarrow \{\psi(\vec{x},t), \psi(\vec{x}',t)\} = 0$$

Let's allow for arbitrary statistics:

$$\phi(x')\phi(x) = (1-\beta)\,\phi(x)\phi(x') \Rightarrow \beta = 0 \text{ is commutator, } \beta = 2 \text{ is anti-com.}$$

$$[\mathcal{O}(x), \mathcal{O}(x')] = \{1-(1-\beta)^4\}\,\phi(x)\phi(x)\phi(x')\phi(x')$$

$\Rightarrow$ must have $\beta = 0$ or 2 for causality

For our K.G. Field, $[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0$

$\Rightarrow$ this is space-like separation: $(t-t)^2 - (\vec{x}-\vec{x}')^2 < 0$

By L.I. of ϕ, $[\phi(x), \phi(x')] = 0$ for $(x-x')^2 < 0$

Do we have a choice for our K.G. Field? Could we have chosen $\{\phi(x), \phi(x')\} = 0$, $\{\Pi(x), \Pi(x')\} = 0$,

$\{\phi(x), \Pi(x')\} = i\delta^{(3)}(\vec{x}-\vec{x}')$ instead?

$\Rightarrow$ can check that this requires

$$\{a(\vec{k}), a(\vec{k}')\} = 0 \qquad \{a^+(\vec{k}), a^+(\vec{k}')\} = 0$$

$$\{a(\vec{k}), a^+(\vec{k}')\} = (2\pi)^3 2\omega \delta^{(3)}(\vec{k}-\vec{k}')$$

Go back to our Hamiltonian

$$H = -\Omega_0 V + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\, \omega \underbrace{\left[a(\vec{k})a^+(\vec{k}) + a^+(\vec{k})a(\vec{k})\right]}_{\{a(\vec{k}), a(\vec{k})\} = (2\pi)^3 2\omega \delta^{(3)}(\vec{k}-\vec{k})}$$

$$= V\left\{\frac{\Lambda^4}{16\pi^2} - \Omega_0\right\}$$

$\Rightarrow$ just a constant; trivial theory

We're locked into our choice of commutation relations. If we had used a Fermionic Lagrangian, we'd need anti-commutators

giving Dirac eq.

This is known as the spin-statistics thm; a non-trivial, causal QFT requires commutation relations requires commutators for integer spin, anti-commutators for half-integer.

Scattering + LSZ reduction

Let's set up scattering experiments, which will typically be what we are interested in.

Vacuum (no particles): $|0\rangle$

Single particle state: $|k\rangle = a^\dagger(\vec{k})|0\rangle$

$$a(\vec{k}) = 0$$

Normalization of our states

$$\langle k|k'\rangle = \langle 0|a(\vec{k})a^\dagger(\vec{k}')|0\rangle = \langle 0|[a(\vec{k}), a^\dagger(\vec{k}')]|0\rangle$$
$$= (2\pi)^3\, 2\omega\, \delta^{(3)}(\vec{k}-\vec{k}')$$

We'll prepare an initial state of two particles (think of two beams colliding) and see how they evolve. We'll initially want them to be spatially separated, so our plane waves won't work $\Rightarrow$ we'd better prepare wave-packets

$$\cancel{a_1^\dagger = \int d^3k\, F_1(\vec{k})\, a^\dagger(\vec{k})}$$
$$\cancel{\text{with } F_1(\vec{k}) = e^{-\frac{(\vec{k}-\vec{k}_1)^2}{4\sigma^2}}}$$