Example: S.H.O.

Let's do the example of a simple harmonic oscillator, with $H = \frac{P^2}{2} + \frac{1}{2}\omega^2 Q^2$ (set $m=1$)

Set $\frac{\partial}{\partial p}\{p\dot{q} - H\} = 0$ to transform to Lagrangian

$$\rightarrow \dot{q} = P$$

$$p\dot{q} - H(1-i\varepsilon) = \dot{q}^2 - \left\{\frac{\dot{q}^2}{2} + \frac{1}{2}\omega^2 q^2\right\}(1-i\varepsilon)$$

$$= \frac{1}{2}\dot{q}^2(1+i\varepsilon) - \frac{1}{2}\omega^2 q^2(1-i\varepsilon)$$

$$\Rightarrow \langle 0|0\rangle_J = \int Dq \; e^{i\int_{-\infty}^{\infty} dt \left\{\frac{1}{2}(1+i\varepsilon)\dot{q}^2 - \frac{1}{2}\omega^2 q^2(1-i\varepsilon) + qJ\right\}}$$

Focus on the integrand in the exponent $\Rightarrow$ Fourier transform

$$q(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt}\tilde{q}(E) \qquad \text{(same for } J\text{)}$$

$$\Rightarrow \dot{q}(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi}(-iE)e^{-iEt}\tilde{q}(E)$$

$$\Rightarrow i\int_{-\infty}^{\infty} dt \frac{dE}{2\pi}\frac{dE'}{2\pi}\left\{-\frac{1}{2}(1+i\varepsilon)EE'\tilde{q}(E)\tilde{q}(E') - \frac{1}{2}\omega^2(1-i\varepsilon)\tilde{q}(E)\tilde{q}(E') + \tilde{J}(E)\tilde{q}(E')\right\} e^{-i(E+E')t}$$

$$= i\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left\{+\frac{1}{2}E^2(1+i\varepsilon)\tilde{q}(E)\tilde{q}(-E) - \frac{1}{2}\omega^2(1-i\varepsilon)\tilde{q}(E)\tilde{q}(-E) + \frac{1}{2}\tilde{J}(E)\tilde{q}(-E) + \frac{1}{2}\tilde{J}(-E)\tilde{q}(E)\right\}$$

done for convenience

This is almost quadratic in $\tilde{q}$; we can get rid of the linear terms by shifting variables in the Functional integration (think of as shifting each of the discrete q_i)

$$\frac{1}{2}\left\{\tilde{q}(E)\tilde{q}(-E)\left[E^2-\omega^2+\overbrace{(E^2+\omega^2)i\varepsilon}^{i\varepsilon \text{ since } E^2+\omega^2>0;}\right]+\tilde{J}(E)\tilde{q}(-E)+\tilde{J}(-E)\tilde{q}(E)\right.$$

note equivalence to just taking $\omega^2\to\omega^2-i\varepsilon$

Set $\tilde{q}(E)=\tilde{q}'(E)-\dfrac{\tilde{J}(E)}{E^2-\omega^2+i\varepsilon}$ $\Rightarrow$ linear shift, so no Jacobian

$$\Rightarrow \frac{1}{2}\left\{\frac{\tilde{q}(E)\tilde{q}(-E)}{E^2-\omega^2+i\varepsilon}-\frac{\tilde{J}(E)\tilde{J}(-E)}{E^2-\omega^2+i\varepsilon}\right\}$$

drop the prime

$$\Rightarrow \langle 0|0\rangle_J = e^{-\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{J}(E)\tilde{J}(-E)}{E^2-\omega^2+i\varepsilon}}\underbrace{\int Dq\, e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{q}(E)\tilde{q}(-E)[E^2-\omega^2+i\varepsilon]}}$$

this is what we get if $J=0$; since $|0\rangle$ is a properly normalized quantum state, this must equal unity

$$\Rightarrow \langle 0|0\rangle_J = e^{-\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{J}(E)\tilde{J}(-E)}{E^2-\omega^2+i\varepsilon}}$$

Go back to time integral by reintroducing

$$\tilde{J}(E) = \int_{-\infty}^{\infty} dt\, e^{iEt} J(t)$$

$$-\frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \frac{e^{iE(t-t')}}{E^2-\omega^2+i\varepsilon} J(t) J(t')$$

$$\equiv \frac{i}{2} \int_{-\infty}^{\infty} dt\, dt'\, J(t)\, G(t-t')\, J(t') \quad \text{with}$$

$$G(t) = -\int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{iEt}}{E^2-\omega^2+i\varepsilon}$$

Let's note something about $G(t)$.

Apply to it the differential operator

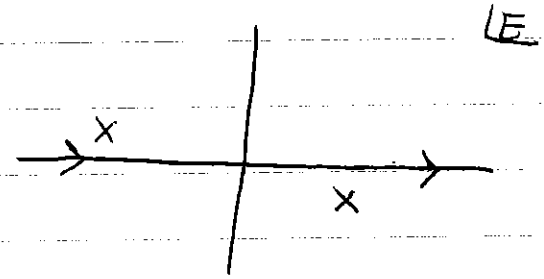
$$\left\{ \frac{\partial^2}{\partial t^2} + \omega^2 \right\} G(t) = -\int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{iEt} \left\{ \frac{-E^2+\omega^2}{E^2-\omega^2} \right\}$$

$$= \delta(t) \quad (\text{drop } +i\varepsilon)$$

$\Rightarrow$ it is the Green's Function for $\frac{\partial^2}{\partial t^2} + \omega^2$

Let's evaluate G explicitly. Because of the $i\varepsilon$, we'd better treat it as a contour integral.

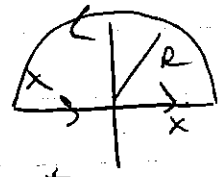
Poles: $E^2 = \omega^2 - i\varepsilon \Rightarrow E = \pm\omega \mp i\varepsilon$



Note that ε is <u>required</u> for this to be well defined.

If $t > 0$: e^{iEt} vanishes for $\mathrm{Im}(t) > 0$

close above



take $R \to \infty$, no contribution from semicircle

Get $2\pi i \times \text{Residue}\Big|_{E=-\omega}$: $\dfrac{i e^{-i\omega t}}{2\omega}$

If $t < 0$: close below, get $-2\pi i \times \text{Residue}\Big|_{E=\omega}$: $\dfrac{i e^{i\omega t}}{2\omega}$

$$\Rightarrow G(t) = \frac{i}{2\omega} e^{-i\omega|t|}$$

Since we have an explicit result for the path integral, we can use it to compute time-ordered matrix elements

$$\langle 0|T\{Q(t_1)Q(t_2)\}|0\rangle = \left\{ \frac{1}{i}\frac{\partial}{\partial J(t_1)} \frac{1}{i}\frac{\partial}{\partial J(t_2)} e^{\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, J(t)\, G(t-t')\, J(t')} \right\}\Bigg|_J$$

$$= \left\{ \frac{1}{i}\frac{\partial}{\partial J(t_1)} \int_{-\infty}^{\infty} dt\, J(t)\, G(t-t_2)\, e^{-\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, J(t)\, G(t-t')\, J(t')} \right\}\Bigg|_J$$

$$= \frac{1}{i} G(t_2 - t_1)$$

This will be our procedure in field theory: use the Green's function of the free theory to compute in perturbation theory time-ordered MEs of the interacting theory

Field Theory path integral

The extension to the Field theory case should be obvious $\Rightarrow Z_0(J) \equiv \langle 0|0\rangle_J = \int D\phi \, e^{i\int d^4x[\mathcal{L}_0 + J\phi]}$

with $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

We implicitly have $m^2 - i\epsilon$ in $\mathcal{L}_0$ to pick out the ground state & define the Green's Function. Go through the same procedure as before to get the Green's Function.

$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\tilde{\phi}(k)$; exponent becomes

$$i\int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\frac{1}{2}\left\{-(k\cdot k'+m^2)\tilde{\phi}(k)\tilde{\phi}(k') + \tilde{J}(k)\tilde{\phi}(k') + \tilde{J}(k')\tilde{\phi}(k)\right\} e^{i(k+k')\cdot x}$$

$$= \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\left\{[k^2-m^2]\tilde{\phi}(k)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)\right\}$$

Set $\tilde{\phi}(k) = \tilde{\phi}'(k) - \frac{\tilde{J}(k)}{k^2-m^2}$

$$\Rightarrow Z_0(J) = e^{-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon}} \underbrace{\int D\phi \, e^{\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{\phi}(k)\tilde{\phi}(-k)[k^2-m^2]}}_{=1 \text{ since } \langle 0|0\rangle = 1 \text{ with } J=0}$$

Go back to position space

$$\Rightarrow \tilde{J}(k) = \int d^4x\, e^{ik\cdot x} J(x)$$

$$\Rightarrow -\frac{i}{2} \int d^4x\, d^4x' \int \frac{d^4k}{(2\pi)^4} J(x) J(x') \frac{e^{ik\cdot(x-x')}}{k^2 - m^2 + i\varepsilon}$$

$$\equiv \frac{i}{2} \int d^4x\, d^4x'\, J(x) \underbrace{\Delta(x-x')}_{} J(x')$$

the Green's Function, called the <u>Feynman propagator</u>

$$\boxed{\Delta(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 - m^2 + i\varepsilon}} \qquad \{\partial^2 + m^2\}\Delta(x) = \delta^{(4)}(x)$$

Note that some texts (Peskin) don't include the overall minus in the definition.

We compute time-ordered products with functional derivatives exactly as before

$$\langle 0|T\{\phi(x_1)\ldots\phi(x_n)\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J(x_n)} Z_0(J)\right]_{J=0}$$

<u>Wick's theorem</u>: $\langle 0|T\{\phi(x_1)\ldots\phi(x_{2n})\}|0\rangle$

$$= \frac{1}{i^n} \sum_{\text{pairings}} \Delta(x_{i_1} - x_{i_2}) \ldots \Delta(x_{i_{2n-1}} - x_{i_{2n}})$$

Adding interactions

Let's deal with a more interesting, non-linear Field theory:
ϕ^3 theory $\Rightarrow$ $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{6}g\phi^3$

Actually, recall those conditions we derived for LSZ Formulae to describe particle scattering $\langle 0|\phi(0)|0\rangle = 0$
$\langle p|\phi(0)|0\rangle = 1$

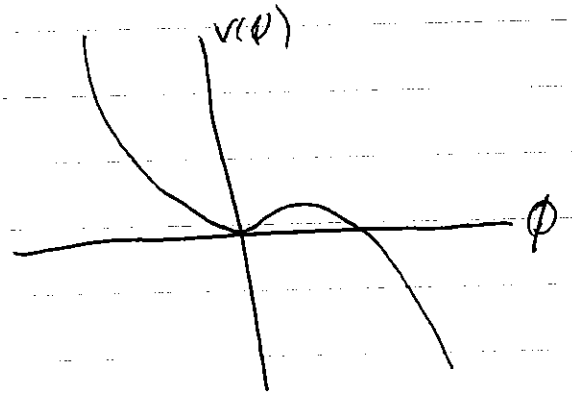
Let's anticipate the need to adjust our theory to make those true

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + \frac{1}{6}Z_g g\phi^3 + Y\phi$$

$\Rightarrow$ roughly, Y will fix $\langle 0|\phi(0)|0\rangle = 0$, Z_ϕ will fix $\langle p|\phi(0)|0\rangle = 1$, & Z_g, Z_m we'll see the need for later (will fix physical mass & cross section)

Note the disclaimer in Srednicki: this theory, in isolation, doesn't have a well-defined ground state. This is clear upon drawing the potential, $V(\phi) = \frac{1}{2}m^2\phi^2 - \frac{1}{6}g\phi^3$

We'll ignore this & use it to practice calculations



Because the eq. of motion for ϕ is non-linear, we can't solve exactly for $G(x)$ & therefore time-ordered products. We'll follow the perturbative method we introduced before

$$Z(J) = \int D\phi \, e^{i\int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + J\phi]}$$

with
$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$
$$\mathcal{L}_1 = \frac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(Z_m - 1)m^2\phi^2 + Y\phi + \frac{1}{6}Z_g g\phi^3$$

$$Z(J) = e^{i\int d^4x \, \mathcal{L}_1\left[\frac{1}{i}\frac{\partial}{\partial J(x)}\right]} \int D\phi \, e^{i\int d^4x \{\mathcal{L}_0 + J\phi\}}$$
$$= \langle 0|0\rangle_J$$

Some notation: $\mathcal{L}_1 = \frac{1}{6}g\phi^3 + \mathcal{L}_{ct}$

$$\mathcal{L}_{ct} = \frac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(Z_m - 1)m^2\phi^2 + Y\phi + \frac{1}{6}(Z_g - 1)g\phi^3$$

is the counterterm Lagrangian, with the parameters fixed by normalization conditions

Now go through the manipulations to write the functional integral in terms of $G(x)$

$$Z(J) = e^{i\int d^4x\, \mathcal{L}_I\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]} e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\,\Delta(x-x')\,J(x')}$$

$$= \langle 0|0\rangle_J \qquad \int D\phi\; e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\,\hat{\phi}(k)[k^2-m^2]\hat{\phi}(-k)}$$

We can't the second line to one anymore; because of $\mathcal{L}_I$, the ground state is changed (derivatives in $\mathcal{L}_I$ can bring down $\Delta(x-x')$ factors that survive the $J\to 0$ limit). However, the $\frac{\delta}{\delta J}$ don't touch this piece; we'll call it C.

$$Z(J) = C\, e^{i\int d^4x\, \mathcal{L}_I\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]} e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\,\Delta(x-x')\,J(x')}$$

we know $Z(0)=1\,(=\langle 0|0\rangle)$, + this will fix C

Ignore $\mathcal{L}_{ct}$ For now, Focus on the ϕ^3 effect

$$\Rightarrow Z_1(J) = C\, e^{\frac{i}{6}g\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3} e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\,\Delta(x-x')\,J(x')}$$

Expand out the exponentials

$$Z_1(J) = C \sum_{v=0} \frac{1}{v!}\left[\frac{ig}{6}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^v$$

$$\sum_{p=0} \frac{1}{p!}\left[\frac{i}{2}\int d^4y\, d^4z\, J(y)\,J(z)\,\Delta(y-z)\right]^p$$

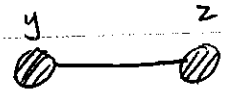
These various terms in the expansion can be interpreted using <u>Feynman diagrams</u>. There are many examples in your book; we'll draw some here.

We'll organize by V

$\boxed{V=0}$: $$C \sum_{p=0} \frac{1}{p!}\left[\frac{i}{2}\int d^4y\, d^4z\, J(y)J(z)\,\Delta(y-z)\right]^p$$

<u>$p=1$</u>: $$\frac{iC}{2}\int d^4y\, d^4z\, J(y)J(z)\,\Delta(y-z)$$

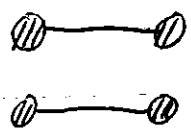
Associate a dark blob to each J, + a line to each propagator:



$\Rightarrow$ this is the diagram corresponding to the contribution

This diagram has a <u>symmetry factor</u>; we can exchange the position of the two sources. This factor is $S=2$, + accounts for the $\frac{1}{2}$ in the denominator

<u>$p=2$</u>: $$C\left(\frac{i}{2}\right)^2\frac{1}{2}\int d^4y\, d^4z\, J(y)J(z)\,\Delta(y-z)$$
$$\int d^4y'\, d^4z'\, J(y')J(z')\,\Delta(y'-z')$$



$\Rightarrow S = 2^2\times 2!$ $\Rightarrow$ interchange blobs on either line (2^2) and interchange the two lines

This diagram is <u>disconnected</u>; if I start at one location I can't get to all others by following a line.

$\boxed{V=1}$ $$\frac{ig}{6}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3 \sum_{P=0} \frac{1}{P!}\left[\frac{i}{2}\int d^4y\, d^4z\, J(y)\, J(z)\, \Delta(y-z)\right]^P$$

Need at least $P=2$ not to vanish.

Note the following general counting arguments. Functional derivatives kill sources, + each <u>vertex</u> (factor of $g\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3$

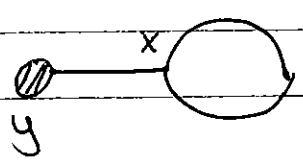
kills 3. Each propagator requires 2 sources.

Total number: $E = 2P - 3V \geq 0 \Rightarrow E =$ <u>external</u> for external sources

<u>$P=2$</u>: $$\frac{ig}{3!}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3 \frac{1}{2!}\left(\frac{i}{2}\right)^2 \int d^4y\, d^4z\, J(y) J(z)\, \Delta(y-z) \int d^4y'\, d^4z'\, J(y') J(z')\, \Delta(y'-z')$$

$E=1$; combinatoric factor of $4\times 3\times 2 = \frac{(2P)!}{(2P-3V)!}$

since the first $\frac{\delta}{\delta J}$ can hit 4 different sources, the next one 3, the next one 2.



Overall phase:

$$i^V \left(\frac{1}{i}\right)^{3V} i^P = i^{P-2V} = 0 \text{ for this diagram}$$

Numerical factor: $$\frac{1}{3!}\frac{1}{2!}\frac{1}{2^2}\, 2!\, 2^2 \times 3 = \frac{1}{2}$$

$\Rightarrow S=2$, can exchange which derivatives acting on the sources give the loop (loop can come from either propagator)

Diagram is <u>connected</u>.

Procedure: Draw all diagrams to given order in g (or V), calculate symmetry factor, Δ for each prop., integrate over all positions

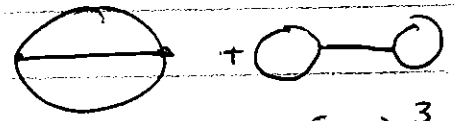
This diagram, for example:

$$\frac{1}{2}\int d^4x\, d^4x\; \Delta(x-y)\,\Delta(x-x)\,J(y)$$

loop

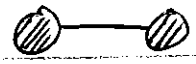
We want to normalize $Z_1(0) = 1$ $\Rightarrow$ let's calculate to see what this is. All functional derivatives must hit a source, so that $J \to 0$ limit doesn't vanish

$\Rightarrow E = 0$, so $2P = 3V$

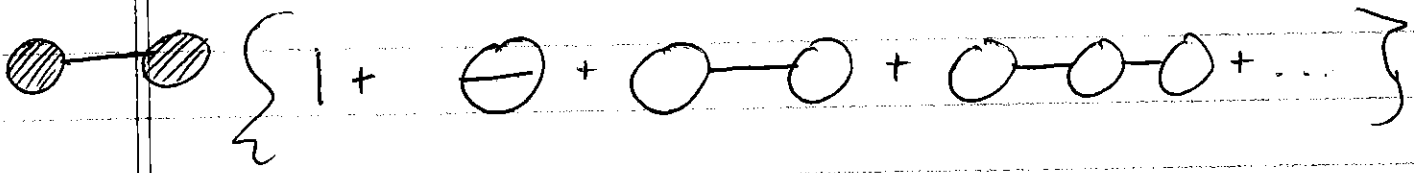
First contribution: $V = 2,\ P = 3$ $\Rightarrow$ 

$S = 2 \times 3!$ $\quad S \to 2^3$

$\Rightarrow$ the sum of all such terms is the sum of all <u>vacuum bubbles</u>. Set $C = \dfrac{1}{Z_1(0)}$, $Z_1(0) = (\sum VB)$

Consider an $E = 2$ contribution such as 

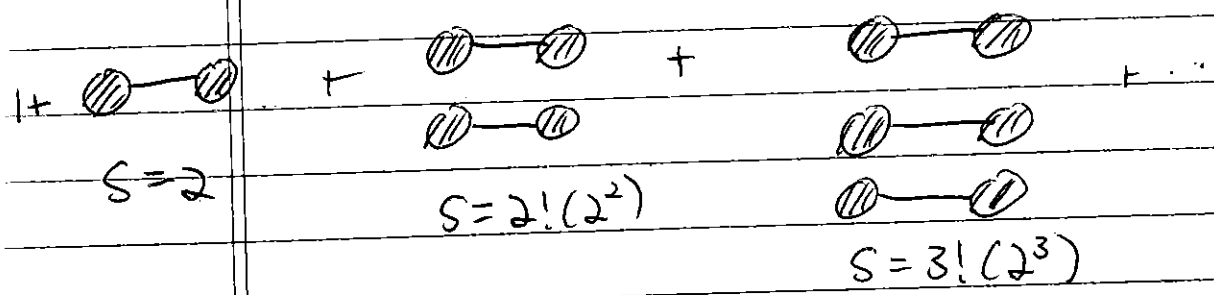
There will be a series of terms in the expansion



$= \{1 + \sum VB\}$ $\Rightarrow$ The $1 + \sum VB$ is divided out by C

$\Rightarrow$ Properly normalized $Z(J)$ has vacuum bubbles removed

There are further simplifications we can make. We will have series of terms of the following forms



⇒ the sum of these exponentiates; the symmetry factors arrange themselves conveniently to do so

$= \exp\left(\bigcirc\!\!-\!\!\bigcirc\right)$ ⇒ the disconnected diagrams are obtained from the connected ones via exponentiation

This is generically true, not just for this particular series. The argument goes as follows.

(1) Let C_I be a connected diagram; a generic diagram is a product of these

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I}$$

n_I = # of times a particular C_I appears

S_D = additional symmetry factor not appearing in C_I. For example, the 2!, 3! above.

(2) S_D only differs from one if the same C_I appears multiple times in a diagram. ⇒ by analogy with above example, $S_D = \prod_I n_I!$ ⇒ $D = \prod_I \frac{1}{n_I!} (C_I)^{n_I}$

calculate $Z_1(J)$ by summing over all diagrams $\Rightarrow$ each diagram labeled by how many of each C_I appear, or equivalently a set of labels $\{n_I\}$

$$Z_1(J) = \sum_{\{n_I\}} D \quad \Rightarrow \text{restriction to no vacuum bubbles understood}$$

$$= \underbrace{\sum_{\{n_I\}}}_{\text{interchange order, allows us to consider each } C_I \text{ separately}} \prod_I \frac{1}{n_I!} (C_I)^{N_I}$$

$$= \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} = \prod_I e^{C_I} = e^{\sum_I C_I}$$

$\Rightarrow Z_1(J) = e^{iW_1(J)}$, with $W_1(J)$ = sum of connected diagrams.

Even more simplifications exist.

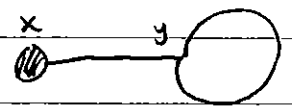
Let's see what imposing our restriction $\langle 0|\phi(x)|0\rangle = 0$ From LSZ reduction leads to.

$$\langle 0|\phi(x)|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta J(x)} Z_1(J)\right]_{J=0} = \left[\frac{\delta}{\delta J(x)} W_1(J)\right]_{J=0}$$

$\Rightarrow$ sum of all diagrams with a single source ($E=1$), with that source removed. Recall $E = 2P - 3V \Rightarrow 1 = 2P - 3V$

<u>$E=1,\ P=1 \Rightarrow V=0$</u> $\Rightarrow$ no diagram exists.

First contribution: $P=1, V=1 \Rightarrow$ only one of $\mathcal{O}(g)$



$$= \frac{ig}{2}\int d^4y\, \frac{1}{i}\Delta(x-y)\,\frac{1}{i}\Delta(y-y)$$

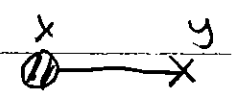
$S=2$

From the rules: ig for each vertex, $\frac{1}{i}\Delta$ for each propagator

This isn't zero; what do we do? Go back and add the counterterm Lagrangian, specifically the $Y\phi$ term.

$$Z_{1+Y}(J) = e^{i\int d^4x\left[\frac{g}{6}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3 + Y\frac{1}{i}\frac{\delta}{\delta J(x)}\right]}\, e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')}$$

Note that the $Y\phi$ term generates a contribution to $\langle 0|\phi(x)|0\rangle \Rightarrow$ $Y\int d^4y\, \Delta(x-y) \Rightarrow$



Sum these terms $\Rightarrow \int d^4y\, \Delta(x-y) \left\{ Y + \frac{g}{2i} \Delta(0) \right\}$

We want this to vanish $\Rightarrow$ choose $Y = -\frac{g}{2i} \Delta(0) \Rightarrow$ this is why we introduced this in the first place!

Let's calculate $\Delta(0) = -\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon}$

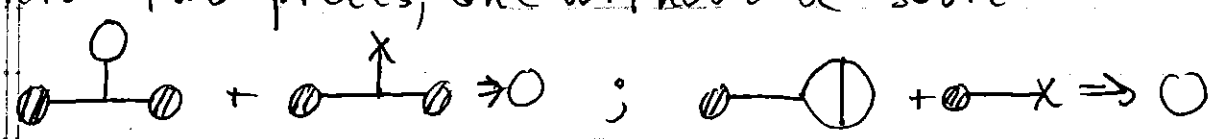
Clearly divergent as $k \to \infty$; regulate by adding the following structure

$$\Delta(0) = -\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon} \left\{ \frac{-\Lambda^2}{k^2 - \Lambda^2 + i\varepsilon} \right\}^2$$

$\Rightarrow$ we don't yet have the machinery to calculate this, but we'll learn later that $\Delta(0) \Rightarrow \frac{-i}{16\pi^2} \Lambda^2$

$\Rightarrow \quad Y = -\frac{g}{32\pi^2} \Lambda^2 \Rightarrow$ infinite counterterm, the first of many...

It turns out that we can choose Y to cancel all such <u>tadpole</u> diagrams: those that, when a line is cut, fall into two pieces, one without a source

$\Rightarrow$ 

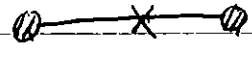
Only one thing left: add the other counter terms

$$\mathcal{L}_{ct} = \frac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(Z_m - 1)m^2\phi^2 + \frac{1}{6}g(Z_g - 1)\phi^3$$

$$Z(J) = e^{\frac{i}{2}\int d^4x \left\{ (Z_\phi - 1)\partial_\mu\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]\partial^\mu\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right] - (Z_m - 1)m^2\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]^2 + \frac{1}{3}g(Z_g - 1)\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]^3 \right\} Z_1(J)}$$

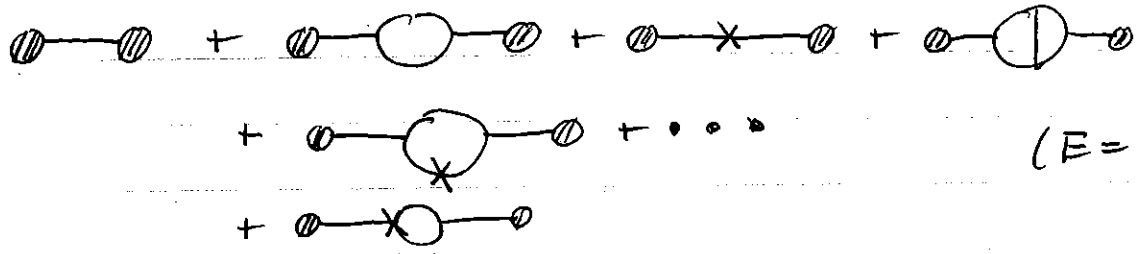
$\Rightarrow$ I.B.P. on First term

$$Z(J) = e^{\frac{i}{2}\int d^4x \left\{ -\frac{1}{i}\frac{\delta}{\delta J(x)}\left[\overset{(Z_\phi - 1)}{\partial^2} + \overset{(Z_m - 1)}{m^2}\right]\frac{1}{i}\frac{\delta}{\delta J(x)} + \frac{1}{3}g(Z_g - 1)\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]^3 \right\} Z_1(J)}$$

The First term leads to a new vertex with two Fields entering:  , For example

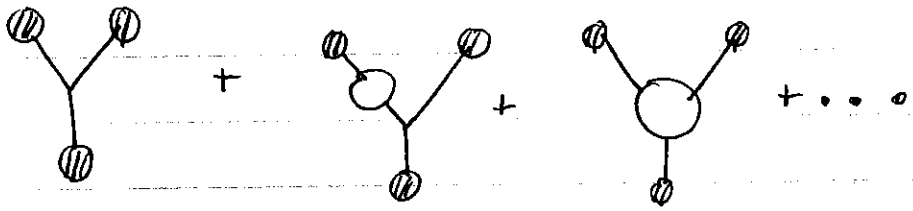
Final restrictions: $Z(J) = e^{iW(J)} \Rightarrow$ no vacuum bubbles, no tadpoles $\Rightarrow E \geq 2$

$\underline{E=2}$:



$(E = 2P - 3V_g - 2V_{ct})$

$\underline{E=3}$:



Scattering amplitudes & Feynman rules

Consider $\langle 0 | T\{\phi(x_1)\phi(x_2)\} | 0 \rangle = \left[\frac{1}{i} \frac{\delta}{\delta J(x_1)} \frac{1}{i} \frac{\delta}{\delta J(x_2)} Z(J) \right]_{J=}$

$$\equiv \partial_1 \partial_2 Z(J) \Big|_{J=0}$$

$$= \partial_1 \left\{ (i\partial_2 W(J)) e^{iW(J)} \right\}_{J=0} = i\partial_1\partial_2 W(J) \Big|_{J=0}$$

$$- \partial_2 W(J) \Big|_{J=0} \partial_1 W(J) \Big|_{J=0}$$ $\Rightarrow$ second term gives tadpoles, & vanishes by $\langle 0 | \phi(x) | 0 \rangle = 0$

What is $i\partial_1\partial_2 W(J)$? $iW(J)$ generates all connected diagrams

$\partial_1\partial_2$ picks those with two sources, removes them, & Labels the endpoints x_1, x_2