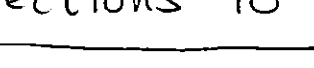
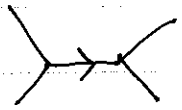
Loop corrections to the propagator

Let's consider the momentum space 2-point function.

Tree-level result:  $= \dfrac{i}{k^2-m^2+i\varepsilon}$

Think of this appearing in the middle of some diagram, e.g.,

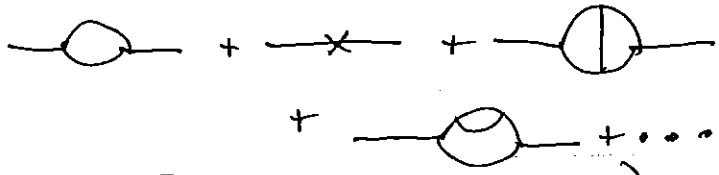


At one-loop, two diagrams appear:

counterterm vertex is

$$-i\left[-(Z_\phi-1)k^2+(Z_m-1)m^2\right]$$

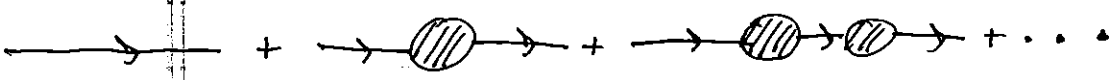
Lump all such diagrams,



$$\equiv \quad \equiv -i\,\Pi(k^2)$$

Such diagrams are called <u>1-particle irreducible</u>

$\Rightarrow$ still connected after a single internal (in a loop) line is cut. We can sum all such corrections to the propagator



$$= \frac{i}{k^2-m^2+i\varepsilon} + \frac{i}{k^2-m^2+i\varepsilon}\,-i\Pi(k^2)\,\frac{i}{k^2-m^2+i\varepsilon} + \cdots$$

$$= \frac{i}{k^2-m^2+i\varepsilon}\left\{1+\frac{\Pi(k^2)}{k^2-m^2+i\varepsilon}+\left[\frac{\Pi(k^2)}{k^2-m^2+i\varepsilon}\right]^2+\cdots\right\} = \frac{i}{k^2-m^2-i\varepsilon-\Pi(k^2)}$$

Before computing $\Pi(k^2)$, let's study a little bit its effect. Recall our LSZ condition $\langle 0|\phi(0)|p\rangle = 1$. Let's apply LSZ to this.

$$\langle 0|\phi(0)|p\rangle = i\int d^4x\, e^{-ip\cdot x}(\partial^2+m^2)\langle 0|T\{\phi(x)\phi(0)\}|0\rangle$$

$$= -i(p^2-m^2)\int d^4x\, e^{-ip\cdot x}\underbrace{\langle 0|T\{\phi(x)\phi(0)\}|0\rangle}_{\text{this is our sum of propagator diagrams in position space}}$$

$$\langle 0|T\{\phi(x)\phi(0)\}|0\rangle = -\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2-m^2+i\epsilon-\Pi(k^2)}$$

At tree level, this becomes

$$\langle 0|\phi(0)|p\rangle = (p^2-m^2)\int\frac{d^4x}{(2\pi)^4}\frac{e^{i(k-p)\cdot x}}{k^2-m^2} = 1, \text{ as required.}$$

with $\Pi(k^2)$, not necessarily true. $\nearrow \left.\frac{d\Pi(k^2)}{dk^2}\right|_{k^2=m^2}$

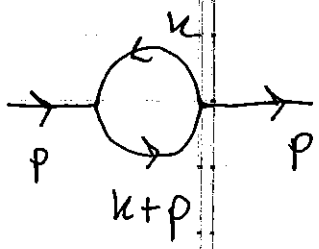
Expand $\Pi(k^2) = \Pi(m^2) + (k^2-m^2)\Pi'(m^2) + \dots$

$$\langle 0|\phi(0)|p\rangle = (p^2-m^2)\int\frac{d^4x}{(2\pi)^4}\frac{e^{i(k-p)\cdot x}}{(k^2-m^2)[1-\Pi'(m^2)]-[\Pi(m^2)]+\dots}$$

$$= \frac{(p^2-m^2)}{(p^2-m^2)[1-\Pi'(m^2)]-[\Pi(m^2)]+(p^2-m^2)^2F(m^2)}$$

For an on-shell state, $p^2=m^2$. Can safely set the $(p^2-m^2)^2 F(m^2)$ to zero in the denominator, but need to be careful with other terms. First, to have the denominator vanish at $p^2=m^2$, need $\Pi(m^2)=0$. Second, to have $\langle 0|\phi(0)|p\rangle=1$, need $\Pi'(m^2)=0$. These are known as the on-shell renormalization conditions, & we will impose them with Z_ϕ, Z_m.

Let's calculate the loop



$$= \frac{1}{2}(ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{[k^2-m^2+i\varepsilon]} \frac{i}{[(k+p)^2-m^2+i\varepsilon]}$$

$$\equiv I$$

Note the following trick, due to Feynman

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\{xA+(1-x)B\}^2} \quad \Rightarrow \text{use to combine denoms.}$$

$$I = \frac{g^2}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\{k^2+2xk\cdot p+xp^2-m^2+i\varepsilon\}^2}$$

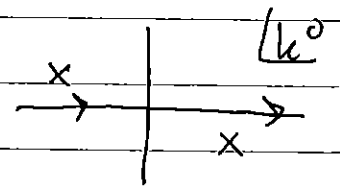
Complete square in denominator by setting $k=k'-xp$

$$\Rightarrow k^2 = k'^2 - 2xk'\cdot p + x^2p^2$$

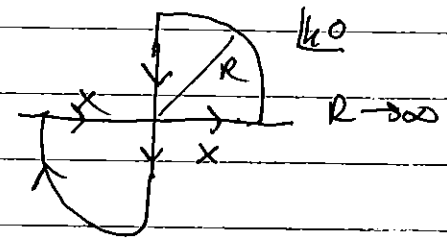
$$k\cdot p = k'\cdot p - xp^2$$

$$I = \frac{g^2}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\{k^2-\Delta\}^2} \quad \text{with } \Delta = m^2 - x(1-x)p^2 - i\varepsilon$$

This looks almost like a spherically symmetric integral, except that $(k^0)^2 - \vec{k}^2 = k^2$. Would be nice to change the sign of the $(k^0)^2$ term. Study k^0 integral in complex plane $(k^0)^2 - \vec{k}^2 - \Delta = 0 \Rightarrow k^0 = \pm\sqrt{\vec{k}^2 + \Delta} \mp i\epsilon$



$\Rightarrow$ close contour:



Encloses no poles, so it vanishes. Assume semi-circular piece vanishes (integrand falls off fast enough at large k^0).

$$\Rightarrow \int_{-\infty}^{\infty} dk^0 + \int_{i\infty}^{-i\infty} dk^0 = 0 \Rightarrow \int_{-\infty}^{\infty} dk^0 = \int_{-i\infty}^{i\infty} dk^0$$

$$I = \frac{g^3}{2}\int_0^1 dx \int_{-i\infty}^{i\infty} dk^0 \int \frac{d^{d-1}k}{(2\pi)^d} \frac{1}{\{(k^0)^2 - \vec{k}^2 - \Delta\}^2}$$

Set $k^0_E = -ik^0 \Rightarrow I = \frac{ig^3}{2}\int_0^1 dx \int_{-\infty}^{\infty} dk^0_E \int \frac{d^{d-1}k}{(2\pi)^d} \frac{1}{\{-(k^0)^2_E - \vec{k}^2 - \Delta\}^2}$

Define $\vec{k}_E = (k^0_E, \vec{k}) \Rightarrow$

$$I = \frac{ig^3}{2}\int_0^1 dx \int \frac{d^d k_E}{(2\pi)^d} \frac{(-1)^{-2}}{\{|\vec{k}_E|^2 + \Delta\}^2}$$

Called a <u>Wick</u> <u>rotation</u>

Focus on the k_E integral; do in spherical coordinates.

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{\{|\vec{k}_E|^2+\Delta\}^2} = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty d|\vec{k}_E| \frac{|\vec{k}_E|^{d-1}}{\{|\vec{k}_E|^2+\Delta\}^2}$$

Ω_d is the solid angle in d-dimensions

Use a trick to get it: $\int d^d x\, e^{-\vec{x}^2} = \pi^{d/2}$ by direct cartesian evaluation

$$\int d^d x\, e^{-\vec{x}^2} = \Omega_d \int_0^\infty d|\vec{x}|\, |\vec{x}|^{d-1} e^{-|\vec{x}|^2} = \frac{\Omega_d}{2}\Gamma\left(\frac{d}{2}\right)$$

$$\Rightarrow \Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \qquad \Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-x}$$ is the Gamma Function

$$\Gamma(n) = (n-1)!\,; \quad \Gamma(-n+x) = \frac{(-1)^n}{n!}\left\{\frac{1}{x} - \gamma + \sum_{k=1}^{n}\frac{1}{k} + \mathcal{O}(x)\right\}$$

Note $\Gamma(0), \Gamma(-n)$ diverges

Euler constant, $\gamma = 0.5772..$

Our k integral becomes

$$\frac{\Omega_d}{(2\pi)^d}\int_0^\infty dy\, \frac{y^{d-1}}{\{y^2+\Delta\}^2}$$

To bring this to standard form, set $x = \frac{\Delta}{y^2+\Delta}$

$$\Rightarrow y^2 = \frac{\Delta(1-x)}{x}$$

$$= \frac{1}{2}\frac{\Omega_d}{(2\pi)^d}\Delta^{d/2-2}\int_0^1 dx\, x^{1-\frac{d}{2}}(1-x)^{d/2-1}$$

The remaining integral is known as a β-Function, and is given by $\int_0^1 dx\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Using this, our k integral becomes

$$\frac{1}{2}\left(\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\right)\frac{\Delta^{\frac{d}{2}-2}}{(2\pi)^d}\frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\Delta^{d/2-2}$$

Combining things, we get

$$\boxed{I = \frac{ig^2}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\int_0^1 dx\,[m^2-x(1-x)p^2-i\varepsilon]^{d/2-2}}$$

For future reference I quote the general result

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{\{k^2-\Delta\}^n} = \frac{i(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\Delta^{d/2-n}$$

Now the Fun begins... Plug in $d=6$, in the numerator get $\Gamma(2-\frac{6}{2}) = \Gamma(-1) \to \infty$ $\Rightarrow$ Need some way to regulate this infinity. There is a very nice way to do this using the Formulae we currently have. Don't set $d=6$; instead set $d=6-\varepsilon$, $\varepsilon \ll 1$ (not to be confused $i\varepsilon$ in denominator.

$$\Gamma\left(2-\frac{(6-\varepsilon)}{2}\right) = \Gamma\left(-1+\frac{\varepsilon}{2}\right) = -\left\{\frac{2}{\varepsilon}-\gamma+1+\mathcal{O}(\varepsilon)\right\}$$

Called <u>dimensional regularization</u> $\Rightarrow$ 1999 Nobel Prize

Go back to I. Note that our coupling constant now has dimensions $[g] = [m]^{\frac{1}{2}(6-d)} = [m]^{\varepsilon/2}$

$\Rightarrow$ replace $g \to g\,\tilde{\mu}^{\varepsilon/2}$, with $[g]=1$ and $[\tilde{\mu}]=[m]$ to keep track of this

$\tilde{\mu}$ is completely arbitrary.

$$I = \frac{ig^2}{2}\frac{1}{(4\pi)^3}\Gamma\left(-1+\frac{\varepsilon}{2}\right)\int_0^1 dx\,\Delta\,\Delta^{-\varepsilon/2}\frac{\tilde{\mu}^{\varepsilon}}{(4\pi)^{-\varepsilon/2}}$$

$$= \frac{-ig^2}{128\pi^3}\left\{\frac{2}{\varepsilon}-\gamma+1\right\}\int_0^1 dx\,\Delta\left[\frac{4\pi\tilde{\mu}^2}{\Delta}\right]^{\varepsilon/2} + \mathcal{O}(\varepsilon)$$

It is conventional at this point to set

$$\tilde{\mu}^2 = \frac{\mu^2 e^{\gamma}}{4\pi}, \text{ so that}$$

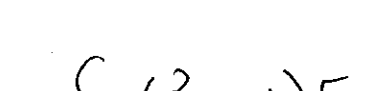
$$I = \frac{-ig^3}{128\pi^3}\left\{\frac{2}{\varepsilon}-\gamma+1\right\}\int_0^1 dx\,\Delta\left[\frac{\mu^2 e^{\gamma}}{\Delta}\right]^{\varepsilon/2}$$

$$= \frac{-ig^2}{128\pi^3}\left\{\left(\frac{2}{\varepsilon}+1\right)\int_0^1 dx\,\Delta \boxed{\cancel{-\gamma\int_0^1 dx\,\Delta}} \boxed{\cancel{+\gamma\int_0^1 dx\,\Delta}} + \int_0^1 dx\,\Delta\,\ln\left[\frac{\mu^2}{\Delta}\right] + \mathcal{O}(\varepsilon)\right\}$$

Plugging back in $\Delta = m^2 - x(1-x)p^2$, and using $\int_0^1 dx\, x(1-x) = \frac{1}{6}$, we have

$$I = \frac{-ig^2}{128\pi^3}\left\{\left(\frac{2}{\varepsilon}+1\right)\left[m^2 - \frac{p^2}{6}\right] - \int_0^1 dx\left[m^2 - x(1-x)p^2\right] \ln\left[\frac{m^2 - x(1-x)p^2}{\mu^2}\right] + O(\varepsilon)\right\}$$

Go back to $\Pi(p^2)$

$-i\Pi(p^2) =$  $\Rightarrow$

$$\Pi(p^2) = -(Z_\phi - 1)p^2 + (Z_m - 1)m^2 + \frac{g^2}{128\pi^3}\left\{\left(\frac{2}{\varepsilon}+1\right)\left[m^2 - \frac{p^2}{6}\right] - \int_0^1 dx\left[m^2 - x(1-x)p^2\right]\ln\left[\frac{m^2 - x(1-x)p^2}{\mu^2}\right]\right\}$$

We need to use Z_ϕ, Z_m to satisfy $\Pi(m^2) = 0$, $\Pi'(m^2) = 0$

(1) $$\Pi'(m^2) = -(Z_\phi - 1) + \frac{g^2}{128\pi^3}\left\{-\frac{1}{6}\left(\frac{2}{\varepsilon}+1\right) + \int_0^1 dx\, x(1-x)\left\{\ln\left[\frac{m^2}{\mu^2}(1-x+x^2)\right]+1\right\}\right\} = 0$$

$\Rightarrow$ solve for $$Z_\phi - 1 = \frac{g^2}{128\pi^3}\left\{-\frac{1}{6}\left(\frac{2}{\varepsilon}+1\right) + \int_0^1 dx\, x(1-x)\left\{\ln\left[\frac{m^2}{\mu^2}(1-x+x^2)\right]+1\right\}\right\}$$

Divergent, but who cares? Plug into $\Pi(p^2)$

$$\Pi(p^2) = (Z_m - 1)m^2 + \frac{g^2}{128\pi^3}\left\{\left(\frac{2}{\varepsilon}+1\right)m^2 - m^2\int_0^1 dx \ln\left[\frac{m^2 - x(1-x)p^2}{\mu^2}\right] + \frac{p^2}{6} + \int_0^1 dx\, x(1-x)p^2 \ln\left[\frac{m^2 - x(1-x)p^2}{m^2(1-x+x^2)}\right]\right.$$

Now set $p^2 = m^2 \Rightarrow$

$$\Pi(m^2) = (Z_m - 1)m^2 + \frac{g^2}{128\pi^3}\left\{\left(\frac{2}{\varepsilon}+1\right)m^2 - m^2\int_0^1 dx \ln\left[\frac{m^2}{\mu^2}(1-x+x^2)\right] + \frac{m^2}{6}\right.$$

$\Rightarrow$ solve for Z_m.

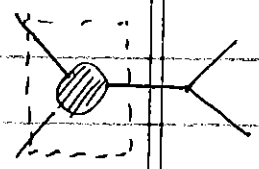
Our "renormalized" $\Pi(p^2)$ is

$$\Pi(p^2) = -\frac{g^2}{128\pi^3}\left\{\int_0^1 dx\,[m^2 - x(1-x)p^2]\ln\left[\frac{m^2 - x(1-x)p^2}{m^2(1-x+x^2)}\right] + \frac{1}{6}(p^2 - m^2)\right\}$$

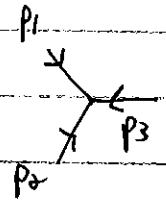
$\Pi(m^2) = 0$, $\Pi'(m^2) = 0$ are called renormalization conditions

Loop corrections to the vertex

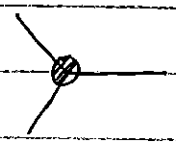
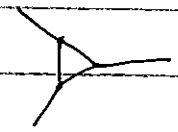
Let's do another example of a loop diagram.

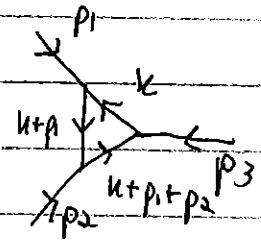


$\Rightarrow$ study corrections to the 3-particle vertex

Tree-level: 

$= ig$

At 1-loop:  $= i(Z_g - 1)g$ 

Focus on the loop 

$$I = (ig)^3 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2} \frac{i}{[(k+p_1)^2-m^2]} \frac{i}{[(k+p_1+p_2)^2-m^2]}$$

Use Feynman's trick to combine denominators:

$$\frac{1}{ABC} = 2\int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3) \frac{1}{\{x_1 A + x_2 B + x_3 C\}^3}$$

$$I = -2g^3 \int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3) \int \frac{d^dk}{(2\pi)^d} \frac{1}{D^3}$$

with $D = k^2 - m^2 + 2(x_1+x_2)\, k\cdot p_1 + 2x_2\, k\cdot p_2 + x_1 p_1^2 + x_2 s$

shift loop momentum: $k = k' - (x_1+x_2)p_1 - x_2 p_2$

$$K^2 = K'^2 - 2(x_1+x_2)K'\cdot p_1 - 2x_2 K'\cdot p_2 + (x_1+x_2)^2 p_1^2 + x_2^2 p_2^2 + 2x_2(x_1+x_2)\, p_1\cdot p_2$$

$$K\cdot p_1 = K'\cdot p_1 - (x_1+x_2)p_1^2 - x_2\, p_1\cdot p_2$$

$$K\cdot p_2 = K'\cdot p_2 - (x_1+x_2)p_1\cdot p_2 - x_2\, p_2^2$$

After this shift, $D = K'^2 - m^2 + x_1 p_1^2 + x_2 s + (x_1+x_2)^2 p_1^2 + x_2^2 p_2^2 + 2x_2(x_1+x_2)\, p_1\cdot p_2 - 2(x_1+x_2)^2 p_1^2 - 2x_2(x_1+x_2)\, p_1\cdot p_2 - 2x_2(x_1+x_2)\, p_1\cdot p_2 - 2x_2^2 p_2^2$

$$= k'^2 - m^2 + \{x_1 - (x_1+x_2)^2\} p_1^2 - x_2^2 p_2^2 + x_2 s - 2x_2(x_1+x_2)\, p_1\cdot p_2$$

Use $s = (p_1+p_2)^2 = 2p_1\cdot p_2 + p_1^2 + p_2^2$

$\Rightarrow D = k'^2 - m^2 + x_2 x_3 s + x_1 x_3 p_1^2 + x_1 x_2 p_2^2$ $\quad$ { Note $s = p_3^2$ }

We now have $I = -2g^3 \int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3) \int \frac{d^dk}{(2\pi)^d} \frac{1}{\{k^2-\Delta\}^3}$

with $\Delta = m^2 - x_2 x_3 s - x_1 x_3 p_1^2 - x_1 x_2 p_2^2$

Use our general result $\int \frac{d^dk}{(2\pi)^d} \frac{1}{\{k^2-\Delta\}^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \Delta^{\frac{d}{2}-n}$

$\Rightarrow n=3$ For us

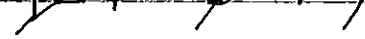
$$I = \frac{ig^3}{(4\pi)^{d/2}} \Gamma(3-\tfrac{d}{2}) \int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3)\, \Delta^{\frac{d}{2}-3}$$

Go to $d = 6-\varepsilon$; add $\tilde{u}^{\varepsilon/2}$ For dimensions

$$\Gamma(3-\tfrac{d}{2}) = \Gamma(\tfrac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$$

$$I = \frac{ig^3}{64\pi^3}\int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3)\left\{\frac{2}{\varepsilon} - \ln\left[\frac{-x_2x_3 s - x_1x_3 p_1^2 - x_1x_2 p_2^2 + m^2}{\mu^2}\right] + \mathcal{O}(\varepsilon)\right\}$$

$$= \frac{ig^3}{64\pi^3}\left\{\frac{1}{\varepsilon} - \int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3)\, \ln[\ldots]\right\}$$

Divergent, so we need to include our $i(Z_g-1)g$ counterterm. How do we choose it? Some freedom here; in QED, we would demand that the theory reproduce an experimentally measured cross section. Here, we will simply demand that the 3-point vertex $iV_3(p_1^2, p_2^2, p_3^2) =$  $+ \ldots$

satisfy $V_3(0,0,0) = g$

$$iV_3(0,0,0) = ig + i(Z_g-1)g + \frac{ig^3}{64\pi^3}\left\{\frac{1}{\varepsilon} - \ln\left[\frac{m^2}{\mu^2}\right]\int_0^1 dx_1 dx_2 dx_3\, \delta(1-\Sigma x_i)\right\}$$

$$= ig + i(Z_g-1)g + \frac{ig^3}{64\pi^3}\left\{\frac{1}{\varepsilon} - \frac{1}{2}\ln\left[\frac{m^2}{\mu^2}\right]\right\} = ig$$

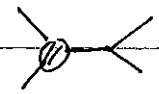
$$\Rightarrow Z_g - 1 = -\frac{g^3}{64\pi^3}\left\{\frac{1}{\varepsilon} - \frac{1}{2}\ln\left[\frac{m^2}{\mu^2}\right]\right\}$$

For arbitrary momenta,

$$V_3(p_1^2, p_2^2, p_3^2) = g - \frac{g^3}{64\pi^3}\int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3)\, \ln\left[\frac{m^2 - x_2x_3 p_3^2 - x_1x_3 p_1^2 - x_1x_2 p_2^2}{m^2}\right]$$

Remember $m^2 \to m^2 - i\varepsilon$ in case argument of logarithm is negative in numerator

Let's evaluate this in an example case: two particles on shell, $p_1^2 = p_2^2 = m^2$, but $p_3 = (p_1+p_2)^2 = s \gg m^2$ This would be the case for a high-energy scattering:



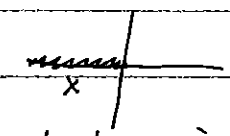
Integral: $\int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3) \ln\left[\frac{-x_2 x_3 s - i\varepsilon}{m^2}\right]$

$\Rightarrow$ note we can remove $-i\varepsilon$ with the understanding $-s \to -s - i\varepsilon$

$$\int_0^1 dx_2 \int_0^{1-x_2} dx_3 \ln\left[-x_2 x_3 \frac{s}{m^2}\right] \cong \frac{1}{2} \ln\left(-\frac{s}{m^2}\right)$$

$$\Rightarrow V_3(p_1^2, p_2^2, p_3^2) = g\left\{1 - \frac{g^2}{128\pi^2} \ln\left[\frac{-s-i\varepsilon}{m^2}\right]\right\}$$

Note: $\ln\left[\frac{-s-i\varepsilon}{m^2}\right] = \ln\left[\frac{|s|}{m^2}\right] - i\pi$



$$\frac{-s}{m^2} - i\varepsilon = \frac{|s|}{m^2} e^{-i\pi}$$

QFT at higher orders always has logarithms of dimensionless ratios