The Lorentz group

We previously found the Lorentz transformation of our scalar field to be $U(\Lambda)\phi(x)U(\Lambda)^{-1} = \phi(\Lambda x)$

$\Rightarrow$ only the space-time is rotated to the new coordinate system, the field doesn't change. From this we derived $U(\Lambda)a(\vec{k})U(\Lambda)^{-1} = a(\Lambda\vec{k})$.

Recall the derivation of this. We can write the two expressions

$$U(\Lambda)\phi(x)U^{-1}(\Lambda) = \phi(\Lambda x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left\{a(\vec{k})e^{-ik\cdot(\Lambda x)} + a^+(\vec{k})e^{ik\cdot(\Lambda x)}\right\}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega}\left\{U(\Lambda)a(\vec{k})U^{-1}(\Lambda)e^{-ik\cdot x} + Ua^+(\vec{k})U^{-1}(\Lambda)e^{ik\cdot x}\right\}$$

Note $k\cdot x = k^\mu x^\nu g_{\mu\nu}$. Set $k^\mu = (\Lambda^{-1})^\mu{}_\rho k'^\rho$

$$\Rightarrow k\cdot x = x_\mu \underbrace{(\Lambda^{-1})^\mu{}_\rho}_{\Lambda_\rho{}^\mu} k'^\rho = k'\cdot(\Lambda x)$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'}\left\{\underbrace{U(\Lambda)a(\Lambda^{-1}\vec{k})U^{-1}(\Lambda)}e^{-ik\cdot(\Lambda x)} + U(\Lambda)a(\Lambda^{-1}\vec{k})U^{-1}(\Lambda)e^{ik\cdot(\Lambda x)}\right.$$

From which we derive $U(\Lambda)a(\vec{k})U^{-1}(\Lambda) = a(\Lambda\vec{k})$

Now consider the transformation of $\partial^\mu\phi(x)$

We can write $\mathcal{U}(\Lambda)\partial^\mu\Phi(x)\mathcal{U}^{-1}(\Lambda) = -i\int\frac{d^3k}{(2\pi)^3 2\omega}\, k^\mu\left\{\mathcal{U}(\Lambda)a(\vec{k})\mathcal{U}^{-1}(\Lambda)e^{-ikx} - \mathcal{U}(\Lambda)a^\dagger(\vec{k})\mathcal{U}^{-1}(\Lambda)e^{ikx}\right\}$

Set $k^\mu = (\Lambda^{-1})^\mu{}_\rho k'^\rho \Rightarrow$

$$= -i\int\frac{d^3k}{(2\pi)^3 2\omega}(\Lambda^{-1})^\mu{}_\rho k^\rho\left\{\mathcal{U}(\Lambda)a(\Lambda^{-1}\vec{k})\mathcal{U}^{-1}(\Lambda)e^{-ik\cdot\Lambda x} - \mathcal{U}(\Lambda)a(\Lambda^{-1}\vec{k})\mathcal{U}^{-1}(\Lambda)e^{ik\cdot\Lambda x}\right\}$$

$$= (\Lambda^{-1})^\mu{}_\rho\,\bar{\partial}^\rho\Phi(\bar{x}) \qquad \text{with } \bar{x}^\mu = \Lambda^\mu{}_\rho x^\rho, \quad \bar{\partial}^\rho = g^{\rho\sigma}\frac{\partial}{\partial\bar{x}^\sigma}$$

To bring this to a more standard form, multiply by $\Lambda^\nu{}_\mu$ and $\mathcal{U}^{-1}(\Lambda)$ on left, $\mathcal{U}(\Lambda)$ on right

$$\mathcal{U}^{-1}(\Lambda)\,\bar{\partial}^\rho\Phi(\bar{x})\,\mathcal{U}(\Lambda) = \Lambda^\nu{}_\mu\partial^\mu\Phi(x) \qquad \text{or (redefine } x\to\Lambda^{-1}x)$$

$$\mathcal{U}^{-1}(\Lambda)\,\partial^\rho(x)\,\mathcal{U}(\Lambda) = \Lambda^\nu{}_\mu\,\tilde{\partial}^\mu\Phi(\Lambda^{-1}x)$$

now, $\bar{x}^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$

This is the template for transformation of a vector field: $\mathcal{U}^{-1}(\Lambda)A^\rho(x)\mathcal{U}(\Lambda) = \Lambda^\rho{}_\sigma A^\sigma(\Lambda^{-1}x)$. Transforms exactly as the index suggests.

We can define a tensor field also.

$$\mathcal{U}^{-1}(\Lambda)\,B^{\rho\sigma}(x)\,\mathcal{U}(\Lambda) = \Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu\,B^{\mu\nu}(\Lambda^{-1}x)$$

However, note the following:

① If $B^{\rho\sigma}$ is symmetric on the left for $\rho\leftrightarrow\sigma$, so is the result on the right.

(2) same if $B^{\rho\sigma}$ is antisymmetric.

(3) Multiply both sides by $g_{\rho\sigma}$

$$\Rightarrow \mathcal{U}^{-1}(\Lambda)\, g_{\rho\sigma} B^{\rho\sigma}(x)\, \mathcal{U}(\Lambda) = \underbrace{g_{\rho\sigma} \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu}}_{g_{\mu\nu}} B^{\mu\nu}(\Lambda^{-1}x)$$

The quantity $g_{\rho\sigma} B^{\rho\sigma}(x)$ transforms as a scalar. The point is that the tensor field representation of the Lorentz group can be broken up into smaller, irreducible representations. What we want to do now is find the irreducible reps. of the Lorentz group. It turns out that there are smaller ones than the vector rep. To find this, we must study the algebra associated with the L.G.

Begin by considering an infinitesimal L.T: $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$

Substitute into $\quad g^{\mu\nu} \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} = g^{\rho\sigma}$

$$\Rightarrow g^{\mu\nu}\{\delta^{\rho}{}_{\mu} + \omega^{\rho}{}_{\mu}\}\{\delta^{\sigma}{}_{\nu} + \omega^{\sigma}{}_{\nu}\} = g^{\rho\sigma}$$

$$= g^{\rho\sigma} + \omega^{\rho\sigma} + \omega^{\sigma\rho} = g^{\rho\sigma} \Rightarrow \omega^{\rho\sigma} = -\omega^{\sigma\rho}$$

L.T. characterized by 6 parameters; 3 boost, 3 rotations. The unitary operator acting on quantum fields for infinitesimal L.T. becomes: $\mathcal{U}(1+\omega) \simeq \mathbb{1} + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}$

$M^{\mu\nu} = -M^{\nu\mu}$ are the 6 generators of the L.G. We want to determine their Lie algebra.

We can derive, - it should be obvious, that

$$\mathcal{U}^{-1}(\Lambda)\, M^{\mu\nu}\, \mathcal{U}(\Lambda) = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}$$

Expand out all quantities to linear order in the small parameter $\omega_{\mu\nu}$:

$$\left\{1-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right\}M^{\mu\nu}\left\{1+\frac{i}{2}\omega_{\gamma\delta}M^{\gamma\delta}\right\}$$

$$=\left\{\delta^{\mu}{}_{\rho}+\omega^{\mu}{}_{\rho}\right\}\left\{\delta^{\nu}{}_{\sigma}+\omega^{\nu}{}_{\sigma}\right\}M^{\rho\sigma}$$

$$\Rightarrow -\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu}+\frac{i}{2}\omega_{\gamma\delta}M^{\mu\nu}M^{\gamma\delta}$$

$$=\omega^{\nu}{}_{\sigma}M^{\mu\sigma}+\omega^{\mu}{}_{\rho}M^{\rho\nu}$$

$$\Rightarrow \frac{i}{2}\omega_{\alpha\beta}[M^{\mu\nu},M^{\alpha\beta}]=g^{\nu\rho}\omega_{\rho\sigma}M^{\mu\sigma}+g^{\mu\sigma}\omega_{\sigma\rho}M^{\rho\nu}$$

$$=g^{\nu\alpha}\omega_{\alpha\beta}M^{\mu\beta}+g^{\mu\alpha}\omega_{\alpha\beta}M^{\beta\nu}$$

$$=\frac{1}{2}\omega_{\alpha\beta}\left\{g^{\nu\alpha}M^{\mu\beta}-g^{\nu\beta}M^{\mu\alpha}+g^{\mu\alpha}M^{\beta\nu}-g^{\mu\beta}M^{\alpha\nu}\right\}$$

we derive $[M^{\mu\nu},M^{\alpha\beta}]=i\left\{g^{\nu\beta}M^{\mu\alpha}+g^{\mu\beta}M^{\alpha\nu}-g^{\nu\alpha}M^{\mu\beta}+g^{\mu\alpha}M^{\beta\nu}\right\}$

We can write these in terms of the rotation boost operators by defining $J_i=\frac{1}{2}\varepsilon_{ijk}M^{jk}$, $K_i=M^{i0}$

Multiply both sides of the commutation relation by

$$\frac{1}{2}\varepsilon_{aij}\frac{1}{2}\varepsilon_{bkl}\Rightarrow[J_a,J_b]=\frac{i}{4}\varepsilon_{aij}\varepsilon_{bkl}\left\{g^{jl}M^{ik}+g^{il}M^{kj}-g^{jk}M^{il}-g^{ik}M^{lj}\right\}$$

Use of the identity $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ gives

$$[J_a, J_b] = i M^{ab} = i\varepsilon_{abc} J_c$$

$$[K_a, K_b] = -i\varepsilon_{abc} J_c$$

$\Rightarrow$ can find these similarly

$$[J_a, K_b] = i\varepsilon_{abc} K_c$$

We can recognize the SU(2) Lie algebra of angular momentum. It turns out we can make this even simpler.

Define $N_i = \frac{1}{2}\{J_i - iK_i\}$ $\Rightarrow$ $J_i = N_i + N_i^+$

$N_i^+ = \frac{1}{2}\{J_i + iK_i\}$ $\quad K_i = i(N_i - N_i^+)$

$$[N_i, N_j] = \frac{1}{4}[J_i - iK_i, J_j - iK_j] = \frac{i}{4}\varepsilon_{ijk}\{J_k + J_k - iK_k - iK_k\}$$
$$= i\varepsilon_{ijk} N_k$$

$$[N_i^+, N_j^+] = i\varepsilon_{ijk} N_k^+$$

$$[N_i, N_j^+] = \frac{1}{4}[J_i - iK_i, J_j + iK_j] = \frac{i}{4}\varepsilon_{ijk}\{J_k - J_k - iK_k + iK_k\}$$
$$= 0$$

The algebra is just $SU(2) \otimes SU(2)$. We know exactly the representation theory from QM. Each SU(2) is labeled by ½-integer values, called j or j'. Number of components are $2j+1$. Label reps as (j, j').

Lowest reps: $(0,0)$ $(1,0)$
$(\frac{1}{2},0)$ $(0,1)$
$(0,\frac{1}{2})$ etc.
$(\frac{1}{2},\frac{1}{2})$

What are the possible ~~spins~~ angular momenta of each rep.? Note
$J_i = N_i + N_i^+ \Rightarrow$ just addition of $\frac{1}{2}$ in QM

$(0,0) \Rightarrow$ scalar

$(\frac{1}{2},0)$ + $(0,\frac{1}{2}) \Rightarrow$ spin-$\frac{1}{2}$, spinor Fields

$(\frac{1}{2},\frac{1}{2}) \Rightarrow$ has $\underbrace{\text{spin-0 + spin-1}}_{\text{4 components total}}$. These are the vector Fields. Time component is spin-0, spatial is spin-1

$(1,0)$ + $(0,1) \Rightarrow$ 3-components each. Can be represented as an anti-symmetric, self-dual tensor $F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$

Maxwell Field strength tensor is $(1,0) \oplus (0,1)$

Note that $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ are related by conjugation

Consider first the $(1/2, 0)$ or left-handed spinor fields. They transform under L.T. as

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a{}^b(\Lambda)\,\psi_b(\Lambda^{-1}x)$$

where $a = 1, 2$. For an infinitesimal transformation,

$$L_a{}^b = \delta_a{}^b + \frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b$$

The S_L must be 2×2 matrices that satisfy the Lorentz algebra:

$$[S_L^{\mu\nu}, S_L^{\rho\sigma}] = i\{g^{\mu\rho}S_L^{\nu\sigma} - g^{\nu\rho}S_L^{\mu\sigma} - g^{\mu\sigma}S_L^{\nu\rho} + g^{\nu\sigma}S_L^{\mu\rho}\}$$

Consider the spatial parts of this: $\mu = i$, $\nu = j$, $\rho = k$, $\sigma = l$

We will show that $S_L^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma_k$

Check $[S_L^{ij}, S_L^{kl}] = \varepsilon^{ijm}\varepsilon^{kln}\underbrace{\left[\frac{\sigma_m}{2}, \frac{\sigma_n}{2}\right]}_{i\varepsilon_{mnr}\frac{\sigma_r}{2}}$

$$= i\left\{\varepsilon^{kli}\frac{\sigma_j}{2} - \varepsilon^{klj}\frac{\sigma_i}{2}\right\} \qquad \text{Replace } \sigma_j = \varepsilon^{jrs}S_L^{rs}$$

$$= i\left\{\varepsilon^{kli}\varepsilon^{jrs}\frac{S_L^{rs}}{2} - \varepsilon^{klj}\varepsilon^{irs}\frac{S_L^{rs}}{2}\right\}$$

Now replace $\varepsilon^{kli}\varepsilon^{jrs} = \begin{vmatrix} \delta^{kj} & \delta^{kr} & \delta^{ks} \\ \delta^{lj} & \delta^{lr} & \delta^{ls} \\ \delta^{ij} & \delta^{ir} & \delta^{is} \end{vmatrix}$

$$= i\delta^{ik} S_L^{jl} - i\delta^{il} S_L^{jk} + i\delta^{jl} S_L^{ik} - i\delta^{jk} S^{il}$$

$\Rightarrow$ these furnish a representation in terms of 2×2 matrices. Recall $J_i = \frac{1}{2}\varepsilon_{ijk} M^{jk}$ for the Lorentz group; S_L^{ij} gives us the angular momentum generator for $(\frac{1}{2}, 0)$ fields. Recall that $J_k = N_k + N_k^\dagger$

$$K_k = i(N_k - N_k^\dagger)$$

and $N_k^\dagger = 0$ for this rep. Therefore, $K_k = iJ_k$

$\Rightarrow S_L^{k0} = K_k$ as before, and $S_L^{k0} = \frac{i}{2}\sigma_k$

An infinitesimal L.T acting on a spinor field becomes $\mathbb{1} + \frac{i}{4}\omega_{ij}\varepsilon^{ijk}\sigma_k + \frac{i}{4}\omega_{i0} i\sigma_i$

Set $\omega_{ij} = \varepsilon_{ijl}\theta_l$, $\omega_{i0} = \beta_i \Rightarrow 1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}$

Now go to the $(0, \frac{1}{2})$ rep. We can get this one by swapping $N_k \to N_k^\dagger$, or equivalently by complex conjugating

$[\psi_a(x)]^\dagger = \psi^\dagger_{\dot{a}}(x) \Rightarrow$ denote this rep by a dot over its indices

To get the representation matrices for $(0, \frac{1}{2})$, note that we can write

$$U(\Lambda)^{-1}\psi^\dagger_{\dot{a}}(x)U(\Lambda) = R_{\dot{a}}{}^{\dot{b}}(\Lambda)\psi^\dagger_{\dot{b}}(x)$$

We can also complex conjugate the $(\frac{1}{2},0)$ transformation law to obtain $U(\Lambda)^{-1}\psi^{+}_{\dot{a}}(x)U(\Lambda) = [L_{\dot{a}}{}^{\dot{b}}(\Lambda)]^{*}\psi^{+}_{\dot{b}}(\Lambda^{-1}x)$

Equate $(L_{\dot{a}}{}^{\dot{b}})^{*} = \delta_{\dot{a}}{}^{\dot{b}} - \frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})^{*}{}_{\dot{a}}{}^{\dot{b}}$

$= R_{\dot{a}}{}^{\dot{b}} = \delta_{\dot{a}}{}^{\dot{b}} + \frac{i}{2}\omega_{\mu\nu}(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}$

$$\Rightarrow (S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -(S_L^{\mu\nu})^{*}{}_{\dot{a}}{}^{\dot{b}}$$

Now that we have the L.T. properties for both spinor fields, we want to start building L.I. Lagrangians to describe their dynamics. We want to do this in the simplest way possible. With vector indices, this was easy because of the relation $\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu}$

This told us that if contracted indices completely, we get a Lorentz scalar. $g_{\mu\nu}$ is an invariant symbol of the L.G. We want something similar for spinor indices. Specifically, we want $L_a{}^c L_b{}^d x_{cd} = x_{ab}$.

Perform an infinitesimal L.T. on x, + see what this condition implies

$$\{\delta_a{}^c + \frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^c\}\{\delta_b{}^d + \frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_b{}^d\}x_{cd} = x_{ab}$$

$$\Rightarrow (S_L^{\mu\nu})_a{}^c x_{cb} = -(S_L^{\mu\nu})_b{}^d x_{ad}$$

Specialize to $(S_L^{\mu\nu})_a{}^c = \frac{1}{2}\varepsilon_{ijk}(\sigma_k)_a{}^c$

$$\Rightarrow (\sigma_k)_a{}^c x_{cb} = -\underbrace{(\sigma_k)_b{}^c}_{(\sigma_k^T)^c{}_b} x_{ac} \quad \Rightarrow \quad \sigma_k x = -x\sigma_k^T$$

What matrix x satisfies this property? It turns out to be simple to find one: $x_{ab} = i(\sigma_2)_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \varepsilon_{ab}$

Check for $k=1,2,3$

$\underline{k=1}$: $\sigma_1\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ while

$$-\varepsilon\sigma_1^T = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\underline{k=2}$: $\sigma_2\varepsilon = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

$$-\varepsilon\sigma_2^T = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$\underline{k=3}$: $\sigma_3\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$-\varepsilon\sigma_3^T = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is our first invariant symbol for spinor indices

Note $\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c$; $\psi^a(x) = \varepsilon^{ab}\psi_b(x)$ $\Rightarrow$ use it to raise, lower spinor indices.

Consider $\psi^a \chi_a = \varepsilon^{ab}\psi_b \chi_a = -\varepsilon^{ba}\chi_a \psi_b = \chi^b \psi_b$

Under L.T, $\psi^a \chi_a \to \varepsilon^{ab} L_b{}^c \psi_c L_a{}^d \chi_d = \varepsilon^{dc}\psi_c \chi_d = \psi^a \chi_a$

$\Rightarrow$ as advertised, gives us a Lorentz scalar.

There is obviously a similar symbol for the $(0,\frac{1}{2})$ rep, $\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}}$

We will need one more invariant symbol, in order to take derivatives of spinor fields.

The other symbol we need is required to write a good kinetic term for Fermionic Fields. Consider the quantity

$$\psi^{\dagger}_{\dot a}\,\bar\sigma^{\mu\dot a a}\,\psi_a \quad \text{with} \quad \bar\sigma^{\mu} = (1, -\vec\sigma)$$

We will show that this transforms as a 4-vector, allowing us to add a ∂_μ to form an invariant kinetic term.

Under L.T.,

$$U(\Lambda)^{-1}[\psi^{\dagger}_{\dot a}\bar\sigma^{\rho\dot a a}\psi_a]U(\Lambda) = \psi^{\dagger}_{\dot b} R_{\dot a}{}^{\dot b}\,\bar\sigma^{\rho\dot a a} L_a{}^{b}\psi_b$$

Consider infinitesimal transformations; the linear term on the RHS of above is $\frac{i}{2}\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^{b}\,\bar\sigma^{\rho\dot b a} + \frac{i}{2}\omega_{\mu\nu}(S_R^{\mu\nu})_{\dot a}{}^{\dot b}\,\bar\sigma^{\rho\dot a b}$

Specialize to spatial transformations, for simplicity

$$\omega_{\mu\nu} \to \omega_{ij} = \varepsilon_{ijk}\,\theta^k$$

$$S_L^{\mu\nu} \to S_L^{ij} = \frac{1}{2}\varepsilon^{ijl}\sigma_l$$

The transformation becomes $\frac{i}{2}\theta^i\{(\sigma^i)_a{}^b\,\bar\sigma^{\rho\dot b a} - (\sigma_i)_{\dot a}{}^{\dot b}\,\bar\sigma^{\rho\dot a b}\}$

$$= \frac{i}{2}(\bar\sigma^\rho\sigma_i - \sigma_i\bar\sigma^\rho)^{\dot b b}\,\theta^i$$

We can simplify this. Note that it vanishes for $\rho=0$.

Set $\rho = l$, $\bar\sigma^\rho \to -\sigma^l$. Get $-\frac{i}{2}\theta^i(\sigma^l\sigma_i - \sigma_i\sigma^l)$

$$= \theta^i\varepsilon^{l}{}_{i}{}^{k}\sigma^k = -\varepsilon_i{}^{lk}\theta^k\sigma^i = \varepsilon^{ilk}\theta^k\sigma^i$$

What do we expect to get? If $\psi^{\dagger}_{\dot a}\bar\sigma^{\mu\dot a a}\psi_a$ transforms as a 4-vector, would expect to find

$$\Lambda^{\rho}{}_{\sigma}\,\bar\sigma^{\sigma}$$

Study this for infinitesimal transformations

$$\Rightarrow \frac{i}{2}\omega_{\mu\nu}(S_V^{\mu\nu})^{\rho}{}_{\sigma}\bar{\sigma}^{\sigma} \Rightarrow \text{note } S_V^{\mu\nu} = \frac{1}{i}[g^{\mu\rho}\delta^{\nu}{}_{\sigma} - g^{\nu\rho}\delta^{\mu}{}_{\sigma}]$$

$$\Rightarrow \frac{1}{2}\varepsilon_{ijk}\theta^k[g^{i\rho}\delta^j{}_{\sigma} - g^{j\rho}\delta^i{}_{\sigma}]\bar{\sigma}^{\sigma}$$

$$= \frac{1}{2}\varepsilon_{ijk}\theta^k[g^{i\rho}\bar{\sigma}^j - g^{j\rho}\bar{\sigma}^i]$$

Note that this vanishes for $\rho=0$. Set $\rho=\ell$

$$= \frac{1}{2}\varepsilon_{ijk}\theta^k[g^{i\ell}\bar{\sigma}^j - g^{j\ell}\bar{\sigma}^i] = -\varepsilon_i{}^{\ell}{}_k\theta^k\bar{\sigma}^i = \varepsilon^{i\ell k}\theta^k\sigma^i$$

Same as above; proves that $\psi^{\dagger}_{\dot{a}}\bar{\sigma}^{\mu\dot{a}a}\psi_a$ transforms as a 4-vector.

To make connection with the normal infinitesimal rotation, note that we've found $\sigma^{\ell} \to \sigma^{\ell} + \varepsilon^{i\ell k}\theta^k\sigma^i$

Can write in matrix notation as

$$\begin{pmatrix}\sigma^1\\ \sigma^2\\ \sigma^3\end{pmatrix} \to \begin{pmatrix}\sigma^1\\ \sigma^2\\ \sigma^3\end{pmatrix} + \begin{pmatrix}0 & -\theta & 0\\ \theta & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma^2\\ \sigma^3\end{pmatrix} \simeq \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

There is a similar symbol $\sigma^{\mu}_{a\dot{a}} = (1,\vec{\sigma})$ that ties together $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ indices in the opposite order.

Let's introduce a few notational issues before continuing. To remove the plethora of spinor indices, we will define

$$\chi\psi \equiv \chi^a\psi_a \;,\quad \chi^{\dagger}\psi^{\dagger} \equiv \chi^{\dagger}_{\dot{a}}\psi^{\dot{a}\dagger}$$

This allows us to write $\chi\psi = \chi^a \psi_a = \varepsilon^{ab}\chi_b \varepsilon_{ac}\psi^c$

$$= \underset{\substack{\uparrow \\ \text{anti-comm.} \\ \text{Fields}}}{-}\psi^c \varepsilon^{ab}\varepsilon_{ac}\chi_b = \psi^c \varepsilon_{ca}\varepsilon^{ab}\chi_b = \psi^a\chi_a = \psi\chi$$

Also, $(\chi\psi)^\dagger = (\chi^a\psi_a)^\dagger = \psi_{\dot a}^\dagger \chi^{\dot a \dagger} = \psi^\dagger\chi^\dagger$

One other thing we will eventually need is $\bar\sigma^{\mu\dot a a}\bar\sigma_\mu^{\dot b b}$
It must be proportional to $\varepsilon^{\dot a\dot b}\varepsilon^{ab}$ (no other alternatives!)
Set $\bar\sigma^{\mu\dot a a}\bar\sigma_\mu^{\dot b b} = \# \, \varepsilon^{\dot a \dot b}\varepsilon^{ab}$. To work out #, set
$\dot a = a = 1, \dot b = b = 2$

$$\Rightarrow \bar\sigma^{\mu 11}\bar\sigma_\mu^{22} = \# = \underbrace{(\bar\sigma^0)^{11}}_{1}\underbrace{(\bar\sigma_0)^{22}}_{1} + \underbrace{(\bar\sigma^3)^{11}}_{1}\underbrace{(\bar\sigma_3)^{22}}_{1} = 2$$

$$\Rightarrow \bar\sigma^{\mu\dot a a}\bar\sigma_\mu^{\dot b b} = 2\varepsilon^{\dot a\dot b}\varepsilon^{ab} \qquad \text{Same for } \sigma^{\mu a\dot a}\sigma_\mu^{b\dot b}$$

We're now ready to write down a Lagrangian for our 2-component field. Try

$$[i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi] \Rightarrow \text{Lorentz invariant; is it real?}$$

$(\bar\sigma^\mu_{a\dot a})^\dagger \rightarrow$ Flip $a \leftrightarrow \dot a$

$$[i\psi^{\dagger\dot a}\bar\sigma^\mu_{\dot a a}\partial_\mu\psi^a]^\dagger = -i\partial_\mu\psi^{\dagger a}(\bar\sigma^\mu_{a\dot a})^*\psi^a$$

Use $(\bar\sigma^\mu_{a\dot a})^* = \bar\sigma^\mu_{\dot a a}$ $\quad = -i\partial_\mu\psi^{\dagger\dot a}\bar\sigma^\mu_{\dot a a}\psi^a \Rightarrow$ I.B.P.

$$= i\psi^{\dagger\dot a}\bar\sigma^\mu_{\dot a a}\partial_\mu\psi^a - i\partial_\mu[\psi^\dagger\bar\sigma^\mu\psi]$$

$\uparrow$ total derivative, doesn't change eqs. of motion.

Start with $S = \underbrace{\int d^4x}_{[m]^4} \; i\psi^\dagger \bar{\sigma}^\mu \underbrace{\partial_\mu}_{[m]} \psi \Rightarrow [\psi] = [m]^{3/2}$

Must keep mass dimension of terms in the Lagrangian to be [4] or less for renormalizability. Can have two fields at most $\Rightarrow \mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} m \psi\psi - \frac{1}{2} m^* \psi^\dagger \psi^\dagger$

This is our free-fermionic Lagrangian. It's easiest to get the equation of motion by varying this w.r.t. $\psi^\dagger$

$$\Rightarrow \frac{\partial S}{\partial \psi^\dagger} = 0 \Rightarrow i\bar{\sigma}^\mu \partial_\mu \psi - m^* \psi^\dagger = 0$$

Note that we can always choose $m^* = m$; if $m = |m| e^{i\alpha}$,
Rescale $\psi \to e^{-i\alpha/2} \hat{\psi}$, $\psi^\dagger \to e^{i\alpha/2} \hat{\psi}^\dagger$ $\Rightarrow$ phase cancels out

With indices written out explicitly, we have

$$i\bar{\sigma}^{\mu \dot{a} b} \partial_\mu \psi_b - m \psi^{\dagger \dot{a}} = 0$$

Take hermition conjugate of this:

$$-i(\bar{\sigma}^{\mu a \dot{b}})^* \partial_\mu \psi^\dagger_{\dot{b}} - m \psi^a = 0$$

$$-i \, \varepsilon_{ba} (\bar{\sigma}^{\mu a \dot{b}})^* \varepsilon_{\dot{b}\dot{a}} \partial_\mu \psi^{\dagger \dot{b}} - m \psi_b = 0$$

We need one more relation to clean this up.

Study $\varepsilon_{ba}\varepsilon_{\dot b\dot a}\bar\sigma^{\mu a\dot a} = (\varepsilon\bar\sigma^{\mu}\varepsilon^{T})_{b\dot b} = (-\varepsilon\bar\sigma^{\mu}\varepsilon)_{b\dot b}$ in matrix notation

$\underline{\mu=0}$: gives $\mathbb{1}$ $\quad\underline{\mu=1}$: $-\varepsilon\bar\sigma^{1}\varepsilon = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$
$= \sigma^{1}$

$\underline{\mu=2}$: $\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = -\sigma^{2} = (\sigma^{2})^{T}$

$\underline{\mu=3}$: $\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \sigma^{3}$

$\Rightarrow$ or, $\varepsilon_{ba}\varepsilon_{\dot b\dot a}\bar\sigma^{\mu a\dot a} = \sigma^{\mu T}_{b\dot b} = \sigma^{\mu}_{\dot b b}$

Use this above; write $(\bar\sigma^{\mu a\dot b})^{*} = \bar\sigma^{\mu\dot b a}$ by hermitcity of Pauli matrices

$\Rightarrow -i\,\varepsilon_{ba}\varepsilon_{\dot b\dot a}\bar\sigma^{\mu\dot b a}\partial_\mu\psi^{+\dot b} - m\psi_b = 0$

$= i\,\varepsilon_{ba}\varepsilon_{\dot a\dot b}\bar\sigma^{\mu b a}\partial_\mu\psi^{+\dot b} - m\psi_b = 0$

$= i\sigma^{\mu}_{b\dot b}\partial_\mu\psi^{+\dot b} - m\psi_b = 0$

Gives us a system of two equations for ψ, ψ^{+}

$$\boxed{\cancel{\begin{pmatrix} i\bar\sigma^{\mu\dot b b}\partial_\mu & -m \\ -m & i\sigma^{\mu}_{b\dot b}\partial_\mu \end{pmatrix}\begin{pmatrix}\psi_b\\ \psi^{+\dot b}\end{pmatrix} = 0}}$$

~~Define~~

$$\begin{pmatrix} -m & i\sigma^{\mu}_{b\dot b}\partial_\mu \\ i\bar\sigma^{\mu\dot b b}\partial_\mu & -m \end{pmatrix}\begin{pmatrix}\psi_b\\ \psi^{+\dot b}\end{pmatrix} = 0$$

Define $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{b\dot{b}} \\ \bar{\sigma}^{\mu\dot{b}b} & 0 \end{pmatrix}$, $\Psi = \begin{pmatrix} \psi_b \\ \psi^{\dagger\dot{b}} \end{pmatrix}$

Can write in 4-component notation

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0$$

This is the <u>Dirac equation</u> for a <u>Majorana Field</u>

Majorana Fields have 2 (complex) degrees of Freedom.
We can define a 4-d.o.F <u>Dirac Field</u> as $\Psi = \begin{pmatrix} \psi_b \\ \chi^{\dagger\dot{b}} \end{pmatrix}$

Our Lagrangian for the Majorana Field is

$$\mathcal{L} = i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2} m \psi_i \psi_i - \frac{1}{2} m \psi_i^\dagger \psi_i^\dagger$$

Suppose we have two Fields, so that $i = 1, 2$. Transform the Fields according to

$$\chi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \Rightarrow \psi_1 = \frac{1}{\sqrt{2}}(\chi + \xi)$$

$$\xi = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \qquad \psi_2 = \frac{1}{i\sqrt{2}}(\chi - \xi)$$

$$\Rightarrow \mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

We can combine these into a 4-component Dirac Field $\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix}$

Let's see how to write the Lagrangian in explicit 4-component notation. Note that $\Psi^\dagger = (\chi_{\dot{a}}, \xi^a)$

We introduced before $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix}$

Multiply $\Psi^\dagger \gamma^0 = (\xi^a, \chi^\dagger_{\dot{a}}) \equiv \bar{\Psi}$

Note $\bar{\Psi}\Psi = \xi^a \chi_a + \chi^\dagger_{\dot{a}} \xi^{\dagger\dot{a}} = \xi\chi + \xi^\dagger\chi^\dagger \Rightarrow$ accounts for the mass terms in the Lagrangian. Also study

$$i\bar{\Psi}\gamma^\mu \partial_\mu \Psi = i(\xi^a, \chi^\dagger_{\dot{a}}) \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$= \underbrace{i\xi^\dagger \sigma^\mu \partial_\mu \xi}_{\substack{\text{upon integration by parts, can write as} \\ i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi \,+\, \text{total derivative,}}} + i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi$$

We can therefore use $\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$ For our Dirac Lagrangian. Equation of motion: $i\gamma^\mu\partial_\mu\Psi - m\Psi = 0$

$$\equiv i\not\partial\Psi - m\Psi = 0$$

To solve this, act on by the operator $(i\not\partial + m)$

$\Rightarrow (i\not\partial + m)(i\not\partial - m)\Psi = 0 \Rightarrow (-\not\partial\not\partial - m^2)\Psi$

Note $\not\partial\not\partial = \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu = \frac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\} = \partial^2$

$\Rightarrow (\partial^2 + m^2)\Psi = 0$

Satisfies the wave eq.; will have a Fourier expansion like the scalar Field. Consider a particular mode

$\Psi(x) = u(p)e^{-ip\cdot x} + v(p)e^{+ip\cdot x}$, where u, v are 4-component spinors.

Act on these with $(i\not\partial - m)$, must have $e^{\pm ip\cdot x}$ modes vanish separately. Derive $(\not p - m)u(p) = 0$

$$(\not p + m)v(p) = 0$$

Each eq. is a system of 2 equations, For each component of u or v. Will have two solutions which we'll label by s. Mode expansion is

$$\Psi(x) = \sum_s \int \tilde{dp}\left[b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)v_s(p)e^{+ip\cdot x}\right]$$

$$\int \frac{d^3p}{(2\pi)^3 2\omega}$$

We will solve these by going to the rest Frame where $\vec{p} = 0$

One can show that the angular momentum operator for the Dirac Field is $S_z = \begin{pmatrix} \frac{\sigma_3}{2} & 0 \\ 0 & \frac{\sigma_3}{2} \end{pmatrix}$, and we will demand our

solutions be the following eigenstates of the spin operator:

$S_z u_\pm(0) = \pm \frac{1}{2} u_\pm(0)$

$S_z v_\pm(0) = \mp \frac{1}{2} v_\pm(0)$

$\Rightarrow$ will later lead to

$[J_z, b_\pm^\dagger(0)] = \pm \frac{1}{2} b_\pm^\dagger(0)$

$[J_z, d_\pm^\dagger(0)] = \pm \frac{1}{2} d_\pm^\dagger(0)$

In the rest Frame, $\not{p} = m\gamma^0 \Rightarrow$ have

$$\begin{pmatrix} -m & 0 & m & 0 \\ 0 & -m & 0 & m \\ m & 0 & -m & 0 \\ 0 & m & 0 & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0 \quad , \quad \begin{pmatrix} +m & 0 & m & 0 \\ 0 & +m & 0 & m \\ m & 0 & +m & 0 \\ 0 & m & 0 & +m \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0$$

Tells us $u_1 = u_3$ $\quad\quad$ $v_1 = -v_3$

$u_2 = u_4$ $\quad\quad$ $v_2 = -v_4$

Consider the u-spinor solutions, $\begin{pmatrix} u_1 \\ u_2 \\ u_1 \\ u_2 \end{pmatrix}$.

$$S_z u = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_1 \\ u_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_1 \\ -u_2 \\ u_1 \\ -u_2 \end{pmatrix} \Rightarrow u_+(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sqrt{m}$$

$$u_-(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \sqrt{m}$$

Similarly, we have $v_+(0) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $v_-(0) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

To get the solution for arbitrary $\vec{p}$, we now boost along the z-axis:

$$u_\pm(\vec{p}) = \exp\{i\eta\,\hat{p}\cdot\vec{K}\}\,u_s(0)$$

$$v_\pm(\vec{p}) = \exp\{i\eta\,\hat{p}\cdot\vec{K}\}\,v_s(0)$$