Homework 2

(1) Begin with the (corrected) result derived in class

$$\langle a'', t'' | a', t' \rangle = \underbrace{\frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}}}_{\substack{N+1\ p_j \text{ integrations} \\ \text{only } N\ a_j \text{ ones}}} \int \prod_{n=1}^{N} \Big[\underbrace{\frac{da_n}{2\pi}}_{\substack{\text{From } p_j \\ \text{integral}}} \sqrt{\frac{2\pi m}{i\delta t}} \Big]^N e^{i\delta t \sum_{j=0}^{N} \frac{m}{2}\left[\frac{a_j}{\delta}\right.}$$

Consider a_1 integral: $\frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}} \int da_1\, e^{\frac{im}{2\delta t}\left[(a_1-a_0)^2+(a_2-a_1)^2\right]}$

Shift $a_1 \to a_1 + \frac{1}{2}(a_0+a_2)$

$$\Rightarrow \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}} \int_{-\infty}^{\infty} da_1\, e^{\frac{im}{\delta t}a_1^2}\, e^{\frac{im}{2\times 2\delta t}(a_2-a_0)^2}$$

$$= \frac{1}{\sqrt{2}}\, e^{\frac{im}{2\times 2\delta t}(a_2-a_0)^2}$$

Now a_2 integral, including this factor

$$\frac{1}{2\pi}\frac{1}{\sqrt{2}}\sqrt{\frac{2\pi m}{i\delta t}} \int da_2\, e^{\frac{im}{2\delta t}\left[(a_3-a_2)^2+\frac{1}{2}(a_2-a_0)^2\right]}$$

Shift $a_2 \to a_2 + \frac{1}{3}[a_0 + 2a_3]$

$$\Rightarrow \frac{1}{2\pi}\frac{1}{\sqrt{2}}\sqrt{\frac{2\pi m}{i\delta t}} \int_{-\infty}^{\infty} da_2\, e^{\frac{3im}{4\delta t}}\, e^{\frac{im}{2\times 3\delta t}(a_3-a_0)^2}$$

$$= \frac{1}{\sqrt{3}}\, e^{\frac{im}{2\times 3\delta t}(a_3-a_0)^2}$$

Pattern clears after a_N integration, have

$$\frac{1}{\sqrt{N+1}}\, e^{\frac{im}{2(N+1)\delta t}(a_{N+1}-a_0)^2}$$

Combine remaining overall factor

$$\Rightarrow \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i(N+1)\delta t}}\; e^{\frac{im}{2(N+1)\delta t}(q_{N+1}-q_0)^2}$$

Identify $(N+1)\delta t = t''-t'$, $q_{N+1}-q_0 = q''-q'$

Note $\left[\frac{mq^2}{t}\right] = [\hbar]$, so we must divide by $\hbar$ in exponent

Units of $\langle q''|q'\rangle$ should be $\frac{1}{\text{length}}$, so should have $\frac{1}{\sqrt{\hbar}}$

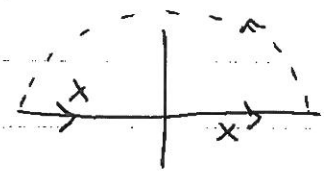
$$\Rightarrow \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\hbar(t''-t')}}\; e^{\frac{im}{2\hbar(t''-t')}(q''-q')^2}$$

(2) Begin with the expression

$$\langle 0|0\rangle_J = e^{-\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{J}(E)\tilde{J}(-E)}{E^2-\omega^2+i\varepsilon}}$$

Pole structure:

$E = \pm|\omega| + i\varepsilon$



Close above; $E = -|\omega|$

$$\langle 0|0\rangle_J = e^{-\frac{\tilde{J}(\omega)\tilde{J}(-\omega)}{4\omega}}$$

$$\Rightarrow P = e^{-\frac{\tilde{J}(\omega)\tilde{J}(-\omega)}{2\omega}}$$

(3) Begin with $Z_0(J) = \int D\phi\, D\phi^+ e^{i\int d^4x \{\partial_\mu\phi\partial^\mu\phi^+ - m^2\phi\phi^+ + J^+\phi + J\phi^+\}}$

Introduce $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\tilde{\phi}(k)$ same for J

$\phi^+(x) = \int \frac{d^4k}{(2\pi)^4} e^{+ik\cdot x}\tilde{\phi}^+(k)$

Exponent of integral becomes

$$i\int d^4x \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4k'}{(2\pi)^4} e^{-i(k-k')\cdot x}\Big[\tilde{\phi}(k)\tilde{\phi}^+(k')\{k\cdot k' - m^2\} + \tilde{J}^+(k')\tilde{\phi}(k) + \tilde{J}(k)\tilde{\phi}^+(k')$$

$$= i\int \frac{d^4k}{(2\pi)^4}\Big\{[k^2-m^2]\tilde{\phi}(k)\tilde{\phi}^+(k) + \tilde{J}^+(k)\tilde{\phi}(k) + \tilde{J}(k)\tilde{\phi}^+(k)\Big\}$$

In the path integral, shift $\tilde{\phi}(k) \to \tilde{\phi}(k) - \frac{\tilde{J}(k)}{k^2-m^2}$

$$\Rightarrow Z_0(J) = e^{-i\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}^+(k)\tilde{J}(k)}{k^2-m^2+i\varepsilon}} \underbrace{\int D\phi\, e^{i\int \frac{d^4k}{(2\pi)^4}\tilde{\phi}^+(k)\tilde{\phi}(k)\times[k^2-m^2]}}_{1}$$

add for pole prescription

Back to position space: $\tilde{J}(k) = \int d^4x\, e^{ik\cdot x} J(x)$

$$\Rightarrow Z_0(J) = e^{+i\int d^4x\, d^4x'\, J(x)\, G(x-x')\, J(x')}$$

with $G(x-x') = -\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{k^2-m^2+i\varepsilon}$

To get a ϕ in time-ordered products, need $\frac{1}{i}\frac{\delta}{\delta J^+}$

For ϕ^+, need $\frac{1}{i}\frac{\delta}{\delta J}$

$$\langle 0|T\{\phi(x_1)\dots\phi(x_n)\,\phi^+(y_1)\dots\phi^+(y_m)\}|0\rangle$$

$$= \left[\frac{1}{i}\frac{\delta}{\delta J^+(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J^+(x_n)}\,\frac{1}{i}\frac{\delta}{\delta J(y_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(y_m)}\,Z_0(J)\right]_{\substack{J=0\\ J^+=0}}$$

Using this, $\langle 0|T\{\phi^+(x_1)\,\phi(x_2)\}|0\rangle = \frac{1}{i}G(x_1-x_2)$

while $\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \langle 0|T\{\phi^+(x_1)\,\phi^+(x_2)\}|0\rangle = 0$

To verify these, we need the mode expansions from HW 1

$$\phi(x) = \int\frac{d^3k}{(2\pi)^3 2\omega}\left\{a(k)e^{-ik\cdot x} + b^+(k)e^{ik\cdot x}\right\}$$

$$\phi^+(x) = \int\frac{d^3k}{(2\pi)^3 2\omega}\left\{b(k)e^{-ik\cdot x} + a^+(k)e^{ik\cdot x}\right\}$$

It is obvious that $\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = 0$; contains only a, b^+, can always move $a(k)$ to the right to annihilate vacuum. Same for $\langle 0|T\{\phi^+(x_1)\phi^+(x_2)\}|0\rangle$

Assume $x_1^0 > x_2^0$

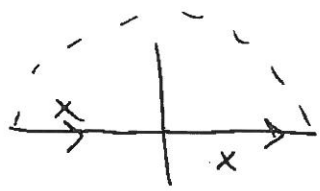
$$\langle 0|\phi^+(x_1)\phi(x_2)|0\rangle = \int\frac{d^3k}{(2\pi)^3 2\omega}\int\frac{d^3k'}{(2\pi)^3 2\omega'}\,\langle 0|\left[b(k)e^{-ik\cdot x_1} + a^+(k)e^{ik\cdot x_1}\right]\left[a(k')e^{-ik'\cdot x_2} + b^+(k')e^{ik'\cdot x_2}\right]|0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \int \frac{d^3k'}{(2\pi)^3 2\omega'} e^{-ik\cdot x_1} e^{ik'\cdot x_2} \langle 0| b(k) b^\dagger(k) |0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik\cdot(x_1-x_2)} \quad \text{with } k^0 = \omega = \sqrt{\vec{k}^2+m^2}$$

This should equal $\frac{1}{i} G(x_1-x_2) = -\frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x_1-x_2)}}{k^2-m^2+i\varepsilon}$

Do k^0 integral



Since $x_1^0 > x_2^0$, must close contour above

$$= -\frac{1}{i} \frac{2\pi i}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{-2\sqrt{\vec{k}^2+m^2}} e^{-i\sqrt{\vec{k}^2+m^2}(x_1^0-x_2^0)} e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}$$

$k^0 = -\sqrt{\vec{k}^2+m^2}$

take $\vec{k} \to -\vec{k}$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik\cdot(x_1-x_2)}$$, as desired

Can similarly check other time ordering

(7) Consider the differential equation

$$\{\partial^2 + m^2\}\, G(x) = \delta^{(4)}(x)$$

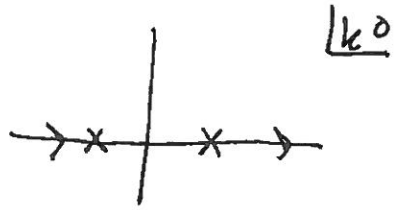
Set $\delta^{(4)}(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}$, $G(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\, \tilde{G}(k)$

$$\Rightarrow \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\{-k^2+m^2\}\, \tilde{G}(k) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}$$

$$\Rightarrow \tilde{G}(k) = \frac{-1}{k^2-m^2} \Rightarrow G(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2-m}$$

There is a problem: denominator vanishes at $k^2 = m^2$

$\Rightarrow k^0 = \pm\sqrt{\vec{k}^2+m^2}$

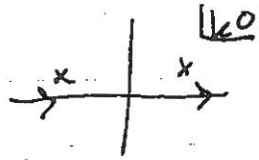


Phase: $e^{ik^0 t}$

If $t>0$, can close contour above, $\mathrm{Im}(k^0)>0$

If $t<0$, can close below

$\Rightarrow$ Shift poles to $k^0 = \pm\sqrt{\vec{k}^2+m^2} + i\varepsilon \Rightarrow$



For $t<0$, close below $\Rightarrow$ vanishes, as desired

Can explicitly do k^0 integral, if desired, to find

$$G_R(x) = -i \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left\{ e^{ik\cdot x} - e^{-ik\cdot x} \right\} \theta(t) \text{ with } k^0 = \sqrt{\vec{k}^2+m}$$

Consider $\{\partial^2+m^2\}\phi(x)=j(x)$ $\Rightarrow$ general solution is a sum of homogeneous + particular solution.
It is simple to construct ϕ_p using the Green's Functio

$\Rightarrow$ $\phi_p(x)=\int d^4y\, j(y)\, G_R(x-y)$ $\Rightarrow$ nothing until $x^0>y^0$

Our homogeneous solution is just what we had in class: $\phi_H(x)=\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left\{a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}\right\}$

Work on ϕ_p: $\phi_p(x)=-i\int d^4y\, j(y)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left\{e^{ik\cdot(x-y)}-e^{-ik\cdot(x-y)}\right\}\Theta(x^0-y^0)$

Set $j(y)=\int\frac{d^4k'}{(2\pi)^4}e^{-ik'\cdot y}\tilde{j}(k')$

$$\Rightarrow \phi_p(x)=-i\int d^4y\int\frac{d^4k'}{(2\pi)^4}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\tilde{j}(k')\,\Theta(x^0-y^0)\left\{e^{ik\cdot(x-y)}e^{-ik'\cdot y}-e^{-ik\cdot(x-y)}e^{-ik'\cdot y}\right\}$$

General solution is $\phi(x)=\phi_H(x)+\phi_p(x)$

Now assume the source was switched on-off in the past $\Rightarrow$ $x^0>y^0$ since $j(x^0,\vec{y})=0$ by assumption $\Rightarrow$ can set $\Theta(x^0-y^0)=1$. Now we can do the y integral in $\phi_p(x)$

$$\phi_p(x) = -i\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left\{\tilde{j}(-k)e^{ik\cdot x} - \tilde{j}(k)e^{-ik\cdot x}\right\}$$

Source is real $\Rightarrow j^+(y) = j(y) \Rightarrow \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot y}\tilde{j}(k)$

$$= \int \frac{d^4k}{(2\pi)^4} e^{+ik\cdot y}\tilde{j}^+(k)$$

$$\Rightarrow \tilde{j}^+(k) = \tilde{j}(-k)$$

$$\Rightarrow \phi_p(x) = -i\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left\{\tilde{j}^+(k)e^{ik\cdot x} - \tilde{j}(k)e^{-ik\cdot x}\right\}$$

Can set $\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left\{e^{-ik\cdot x}\left[a(k) + i\tilde{j}(k)\right] + e^{ik\cdot x}\left[a^+(k) - i\tilde{j}^+(k)\right]\right\}$

It is clear now how to get the Hamiltonian $\Rightarrow$ just shift $a \to a + i\tilde{j}$, $a^+ \to a^+ - i\tilde{j}^+$ in previous result

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\left[a^+(k) - i\tilde{j}^+(k)\right]\left[a(k) + i\tilde{j}(k)\right]$$

Vacuum: $\langle 0|H|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega}\,\omega\,\tilde{j}^+(k)\tilde{j}(k)$

$\Rightarrow \tilde{j}^+\tilde{j}$ produces particles $\Rightarrow$

number of particles produced is

$$N = \int \frac{d^3k}{(2\pi)^3 2\omega}\tilde{j}^+(k)\tilde{j}(k) = \int \frac{d^3k}{(2\pi)^3}\underbrace{\frac{dN}{d^3k}}_{\frac{|\tilde{j}(k)|^2}{2\omega}}$$