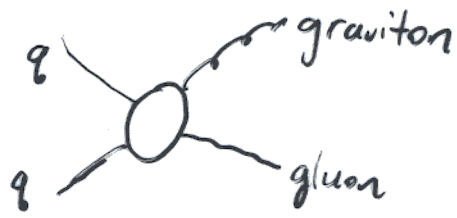
Summing KK modes

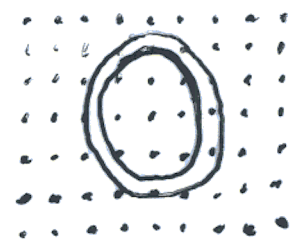
A scattering process for a particular external KK mode with mass m $(= |n|R^{-1})$ is

$$\frac{d\sigma_m}{dt} \sim \frac{1}{M_P^2}.$$



However, need to sum over large multiplicity of KK modes. Picture a n-dimensional m-space lattice, with a particle at each point

$$(n_1 R^{-1}, n_2 R^{-1}, \cdots, n_n R^{-1}).$$



Number of states between m and $m + dm$ is

$$dN \qquad \underbrace{(S_{n-1} m^{n-1} dm)}_{\text{Volume of slice}} R^n$$

density of states

$$\frac{M_P^2}{M_D^{2+n}} S_{n-1} m^{n-1} dm,$$

where $M_P^2 = (RM_D)^n M_D^2$ is used

Therefore,

$$\sum_{\text{KK modes}} \frac{d\sigma_m}{dt} \longrightarrow \frac{M_P^2}{M_D^{2+n}} \int dm\, m^{n-1} S_{n-1} \frac{d\sigma_m}{dt} \simeq \frac{s^{n/2}}{M_D^{2+n}}$$

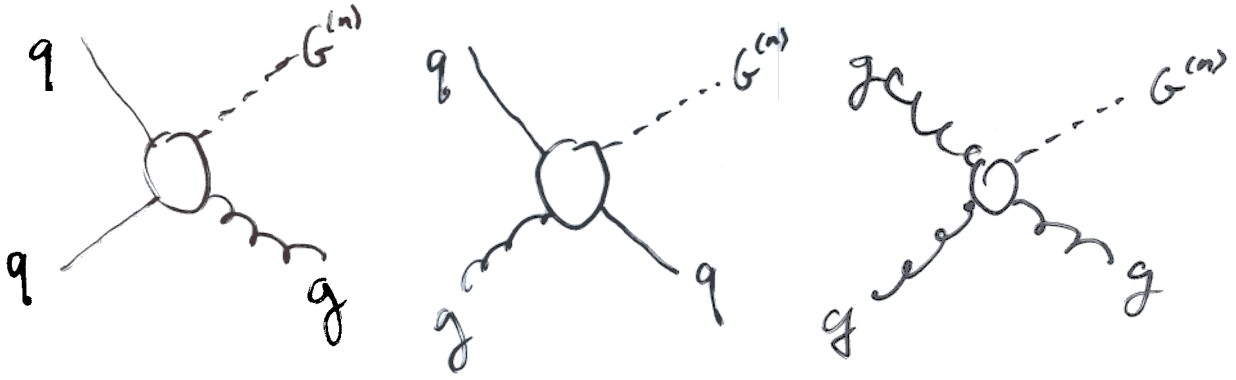
Another approximation to summing up modes is

$$\Delta N_E \sim (ER)^n \sim \frac{M_P^2}{M_D^{2+n}} E^n$$

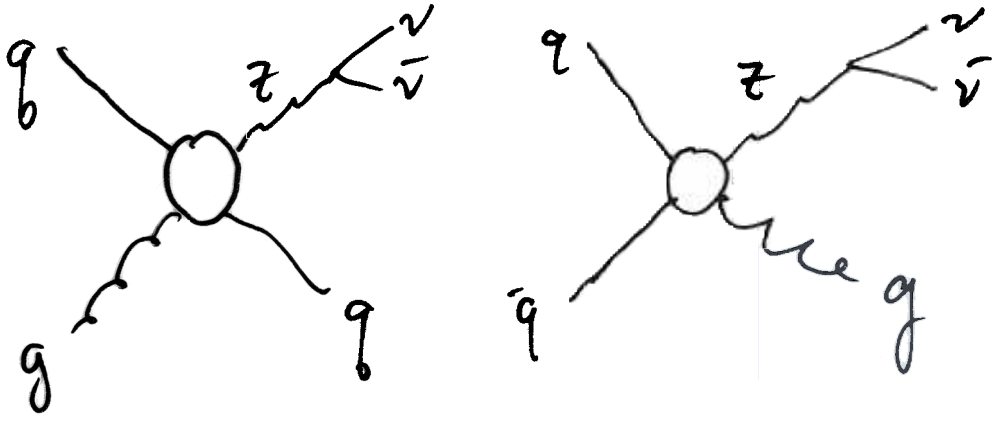
E ~ accessible energy in process

Jet signal and background at Hadron colliders

Study the signal associated with graviton plus jet final states:



The main background is from $pp, p\bar{p} \to$ jet Z, where $Z \to \bar{\nu}\nu$.



Dimension-eight operators

Dimension-eight operators are induced by tree-level exchange of a virtual graviton.

$$\mathcal{A} = \frac{1}{M_P^2} \sum_n T_{\mu\nu} \frac{P^{\mu\nu\alpha\beta}}{s - m^2} T_{\alpha\beta} = \mathcal{S}(s)\, T^{\mu\nu} T_{\mu\nu}$$

$$\mathcal{S}(s) \equiv \frac{1}{M_P^2} \sum_n \frac{1}{s - m^2}$$

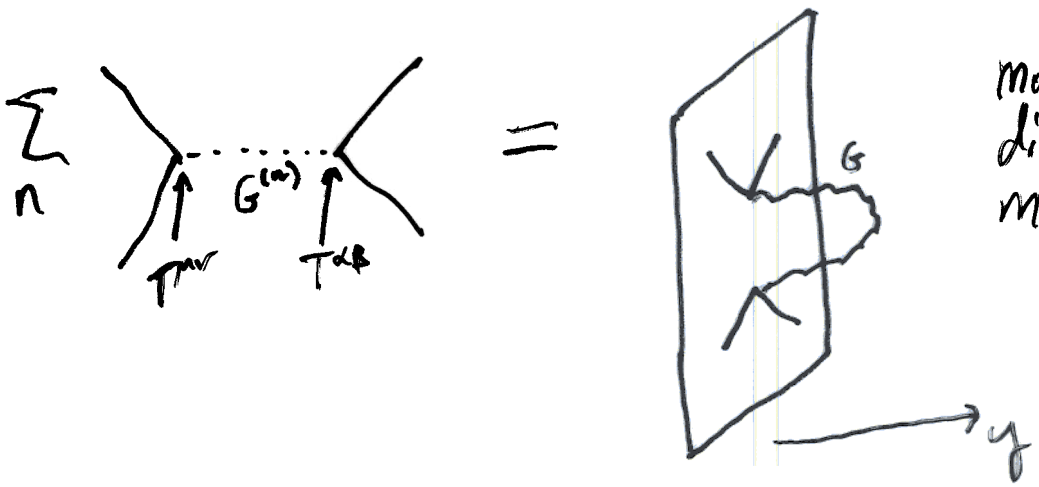
We can calculate $\mathcal{S}(s)$ in the continuous limit ($\Delta m \ll \Delta E$)

$$\mathcal{S}(s) = \frac{1}{M_D^{2+n}} \int d^n q_T \frac{1}{s - q_T^2} = \frac{c_1}{M_D^{2+n}} \frac{S_{n-1}}{2} \Lambda^{n-2} +$$

So, the operators under consideration will be

$$\mathcal{A} = \frac{S_{n-1}}{2}\, c_1\, \frac{\Lambda^{n-2}}{M_D^{n+2}}\, T^{\mu\nu} T_{\mu\nu} = \frac{4\pi}{\Lambda_T^4}\, T^{\mu\nu} T_{\mu\nu}$$

$\Lambda_T \simeq M_D$ is reasonable for all n, but the magnitude is not calculable.



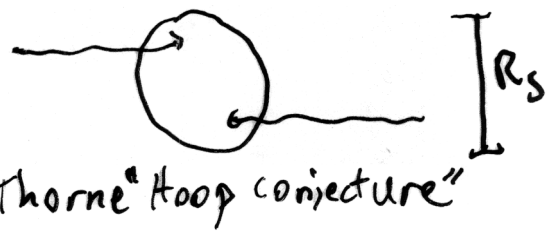
momentum in y direction looks like mass on 3-brane

What happens at $\sqrt{s} \gg M_D$?

One thing that happens is black hole can form (Giddings, Thomas, Dimopoulos, ...)

Naive view: Mass/energy clustered below corresponding Schwarzschild radius R_S gravitationally collapses with probability 1. Colliding partons with c.m. energy $\sqrt{\hat{s}}$ and impact parameter $b < R_S$ form black hole.

$$\sigma_{\text{black hole}} \simeq \pi R_S^2(\hat{s}).$$



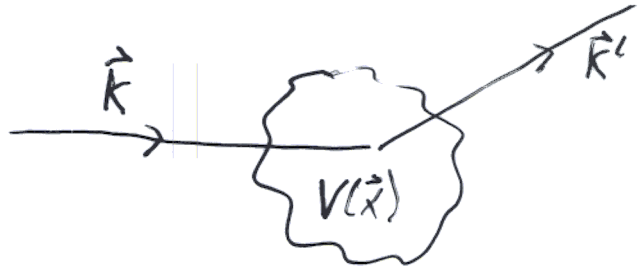
Since

$$R_S \sim \frac{1}{M_D}\left(\frac{\sqrt{\hat{s}}}{M_D}\right)^{\frac{1}{n+1}}$$

we should expect large cross-sections $\sigma \gtrsim 1/M_D^2$ for $\hat{s} > M_D^2$.

This simple argument is more correct as $\hat{s} \gg M_D^2$.

Interested in a more calculable observable. Look to forward scattering.



Quantum Mechanical Scattering

Integral equation for outgoing wave-function $\psi(\vec{x})$ scattering in presence of finite potential $V(\vec{x})$:

$$\psi(\vec{x}) = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} - 2m\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} V(\vec{x}')\psi(\vec{x}')$$

At large $|\vec{x}| - r$ it becomes

$$\psi(\vec{x}) = \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} A(\vec{k}', \vec{k})\right]$$

where

$$A(\vec{k}', \vec{k}) \sim m\int d^3x' e^{i\vec{k}'\cdot\vec{x}'} V(\vec{x}')\psi(\vec{x}')$$

Differential scattering cross-section into solid angle is

$$\frac{d\sigma}{d\Omega} \quad A(\vec{k}, \vec{k}')|^2$$

Born Approximation: Weak scattering potential. Solve integral equation iteratively in expansion about small V. First term (Born term) is replacing $\psi(\vec{x}')$ with plane wave solution in $A(\vec{k}, \vec{k}')$.

For $V(\vec{x}) = V(r)$ then

$$A_{\text{Born}}(\theta) \sim \frac{m}{q}\int_0^\infty dr\, rV(r)\sin qr$$

where $q = |\vec{k} - \vec{k}'| = 2k\sin\theta/2$.

Perturbative expansion
n small V.

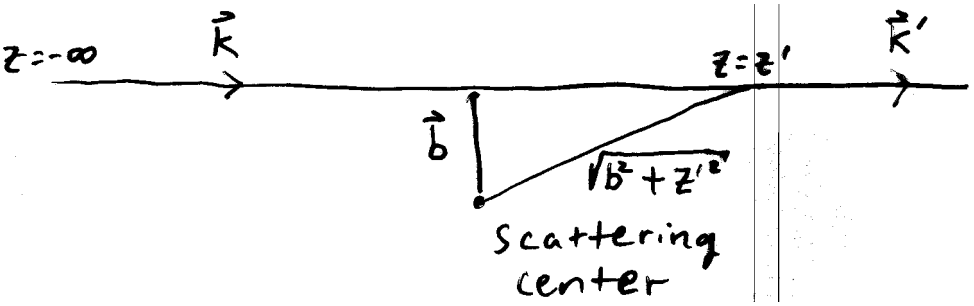
Eikonal Approximation

Change in V over de Broglie wavelength should be small compared to energy. $E \gg V|$ works. (WKB applied to scattering.)

Forward scattering amplitude exactly calculable in approximation. (Equivalent to resumming perturbative expansion to all orders at small angle scattering.)

Identify $\psi \sim e^{i\phi(\vec{x})}$ as wavefunction with $\nabla\phi$ as semi-classical trajectory

Schrödinger equation: $(\nabla\phi)^2 + 2mV(\vec{x}) = k^2$



At impact parameter b the phase is

$$\phi(\vec{x}) = kz - \frac{m}{k}\int_{-\infty}^{z} V(\sqrt{b^2+z'^2})dz' \Longrightarrow \psi(\vec{x}) \sim e^{ikz}e^{i\chi(b,z}$$

where

$$\chi(b,z) = -\frac{m}{k}\int_{-\infty}^{z} dz' V(\sqrt{b^2+z'^2})$$

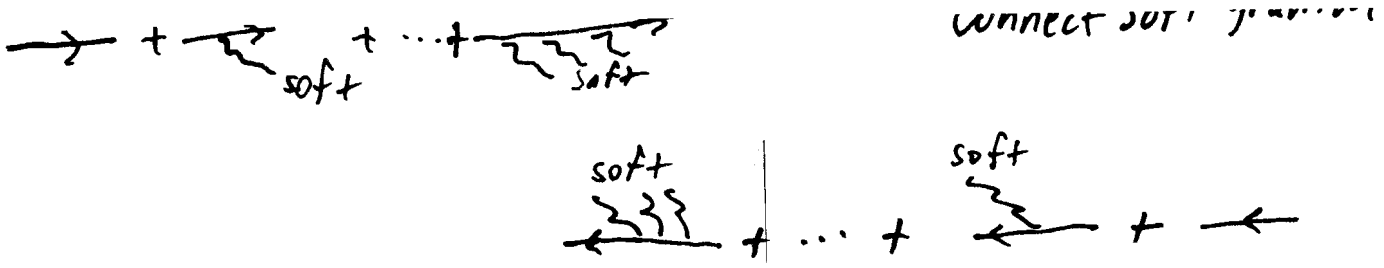
"Eikonal phase"

Then after some algebra (Plug this $\psi(\vec{x})$ into integral eq.)

$$\boxed{A(\vec{k},\vec{k}') = -ik\int_0^\infty db\, bJ_0(qb)\left[e^{i\chi(b,\infty)} - 1\right]}$$ where $q = |\vec{k} - \vec{k}'| \simeq k\theta$.

small momentum transfer (small θ)

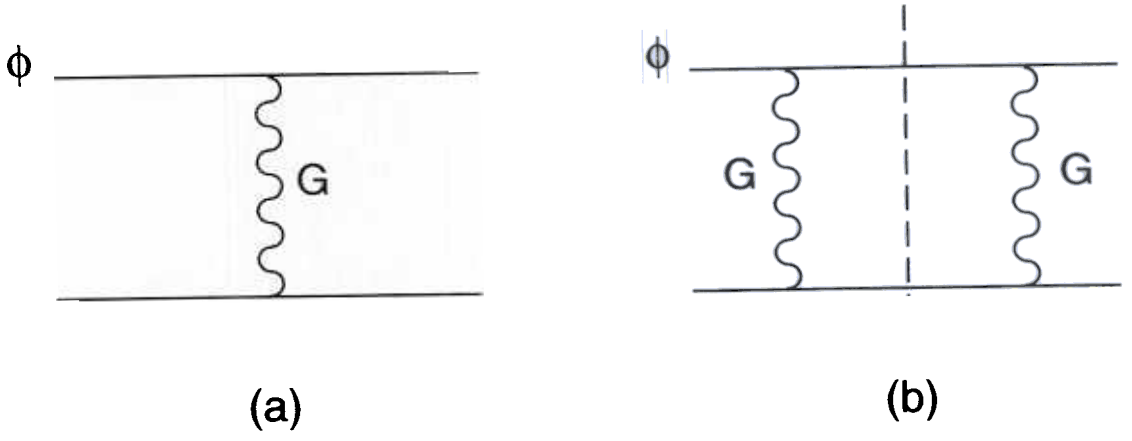
Valid for $\vec{k} \simeq \vec{k}'$ (small angle)



Forward scattering at $\sqrt{s} \gg M_D$

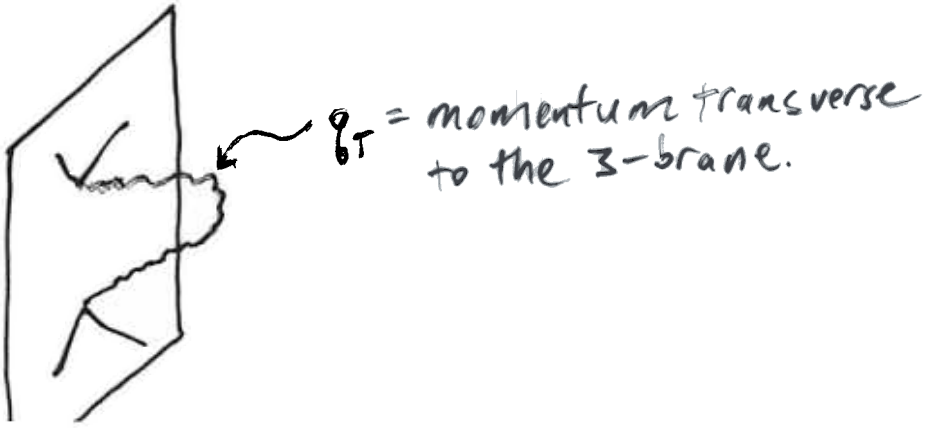
The leading behavior of the forward scattering amplitude is summable to all orders.

Technique of resummation is the "eikonal approximation", where ladder diagrams are summed in the limit of small momentum transfer.



$$\mathcal{A}_{\text{Born}}(-t) = \frac{s^2}{M_D^{n+2}} \int \frac{d^n q_T}{t - q_T^2} = \pi^{\frac{n}{2}} \Gamma(1 + n/2) \left(\frac{-t}{M_D^2}\right)^{\frac{n}{2}-1} \left(\frac{s}{M_D^2}\right)^2$$

This Born amplitude is divergent from the sum over infinite momentum in the transverse (extra-dimensional) space.



Eikonal Resummation

Emparan, Masip, Rattazzi

$$\mathcal{A}_{\rm Born}(q_\perp^2) + \mathcal{A}_{1-\rm loop}(q_\perp^2) + \ldots = -2is \int d^2b_\perp e^{iq_\perp b_\perp} \left(i\chi - \frac{1}{2}\chi^2 + \ldots \right)$$

$$\text{where} \quad \chi(b_\perp) = \frac{1}{2s} \int \frac{d^2q_\perp}{(2\pi)^2} e^{-iq_\perp b_\perp} \mathcal{A}_{\rm Born}(q_\perp^2)$$

Following the combinatorics, one can resum all terms to get

$$\mathcal{A}_{\rm eik} = -2is \int d^2b_\perp e^{iq_\perp b_\perp} \left(e^{i\chi} - 1 \right)$$

Eikonal scattering phase is just the 2-d impact parameter Fourier transform of the Born amplitude.

Direct computation of χ yields

$$\chi = \frac{\pi^{\frac{n}{2}-1}\Gamma(1-n/2)s}{4M_D^{n+2}} \int_0^\infty dq q^{n-1} J_0(qb) = \left(\frac{b_c}{b}\right)^n ,$$

$$\text{where} \quad b_c \equiv \left[\frac{(4\pi)^{\frac{n}{2}-1} s \Gamma(n/2)}{2M_D^{n+2}} \right]^{1/n}$$

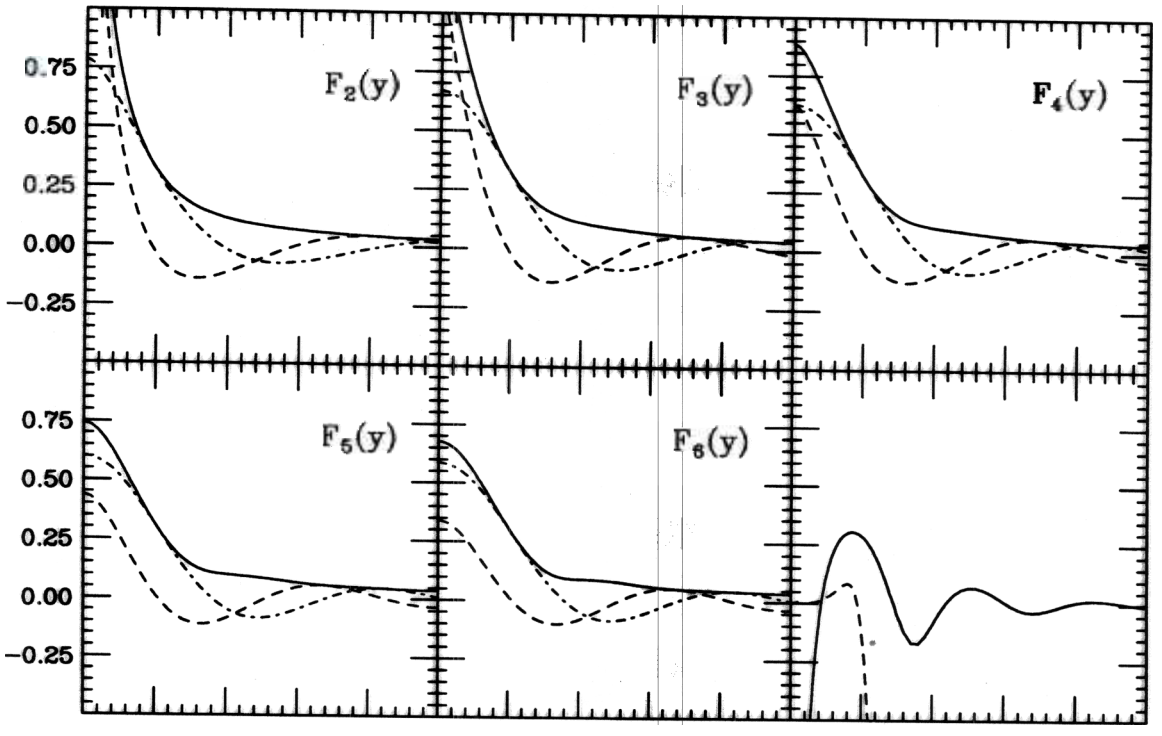
The divergence in $\mathcal{A}_{\rm Born}$ does not survive in χ, except the limit of $b \to 0$ (point-like interactions) smoothly integrated over in the amplitude.

Eikonal Amplitude

The finite eikonal amplitude for scattering the limit of $q^2 \ll$ and $M_D^2 \ll s$

$$\mathcal{A}_{\text{eik}} - 4\pi s b \; F\,(b_c q)$$

where $F_n(y) = -\int_0^\infty dx x J_0(xy)$



Two important scale lengths

In addition to M_D (or equivalently $G_D \sim 1/M_D^{2+n}$) there are two other important scales we have encountered.

$$R_S = \frac{1}{\sqrt{\pi}} \left[\frac{8\Gamma\left(\frac{n+3}{2}\right)}{(n+2)} \right]^{\frac{1}{n+1}} \left(\frac{G_D \sqrt{s}}{c^4} \right)^{\frac{1}{n+1}}$$

$$b_c \propto \left(\frac{G_D s}{\hbar c^5} \right)^{\frac{1}{n}}$$

b_c is a new scale with no 4d analog. As $\hbar \to 0$ (classical limit), $b_c \to \infty$. We will see interesting quantum effects arise for $b \gtrsim b_c$.

Example numbers:

$\sqrt{s} = 10\,\text{TeV}$, $M_D = 2\,\text{TeV}$, $n = 2$: $R_S^{-1} = 1.3\,\text{TeV}$, $b_c^{-1} = 0.6\,\text{TeV}$
$\sqrt{s} = 10\,\text{TeV}$, $M_D = 2\,\text{TeV}$, $n = 6$: $R_S^{-1} = 0.7\,\text{TeV}$, $b_c^{-1} = 0.5\,\text{TeV}$

Corrections to the Eikonal Amplitude

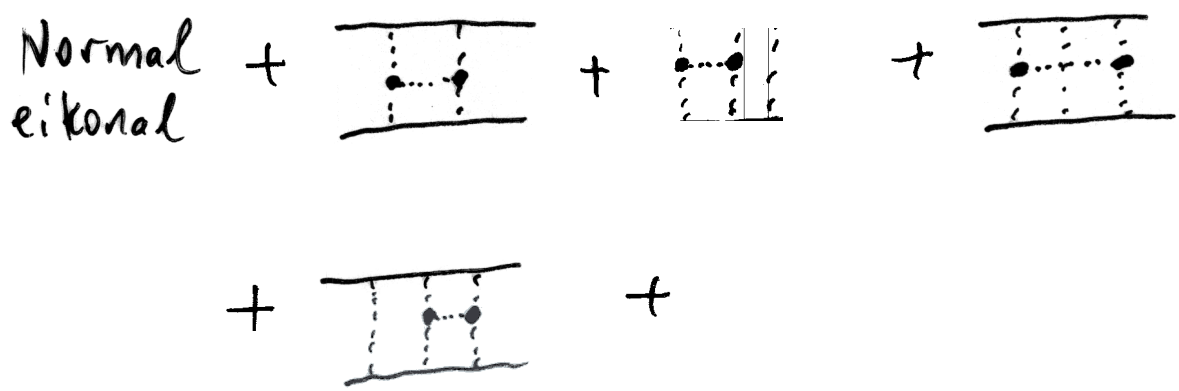
Following Amati et al., estimate corrections by improving eikonal resummation with inclusion of two-loop "H diagrams", leading to

$$\chi \quad \rightarrow \quad \left(\frac{b_c}{b}\right)^n \left[1 + \mathcal{O}\left(\frac{R_S}{b}\right)^{2n+2}\right]$$

This translates to the following corrections in the amplitude

$$\mathcal{O}\left(\frac{t}{s}\right) + \mathcal{O}\left(\frac{M_D^2}{s}\right)^{1+\frac{2}{n}}$$

Small angle scattering and transplanckianality keep corrections under control.



Physical intuition on corrections

Eikonal best when $R_s \ll 1$ (Gravity potential not overwhelming part. E)

$$\left[\frac{q}{\sqrt{s}} \ll 1 \quad \frac{M_D^2}{s} \ll 1\right]$$

b

corrections $\sim \left(\frac{R_s}{b}\right)^p$

Dictionary

$$M_D\left(\frac{s}{M_D^2}\right)^n \qquad b_s \qquad M_D\left(\frac{q}{\sqrt{s}}\right)^{n+1}\left(\frac{s}{M_D^2}\right)^{2(n+1)} \qquad b_c\left(\frac{n}{qb_c}\right)^{\frac{1}{n+1}}$$

$$M_D\left(\frac{s}{M_D^2}\right)^{2(n+1)}$$

(If $qb_c \gg 1$ then A_{eik} dominated at b b_s

$$\left(\frac{R_s}{b}\right)^p \to \left(\frac{R_s}{b_s}\right) \qquad \left(\frac{q}{\sqrt{s}}\right)^{\frac{p}{n+1}}$$

q $\sqrt{s}$ small angle makes this small

(i) If $qb \ll 1$ then A_{eik} dominated at $b \sim b_c$

$$\left(\frac{R_s}{b}\right)^p \to \left(\frac{R_s}{b_c}\right)^p \qquad \left(\frac{M_D^2}{s}\right)^{\frac{(n+2)p}{2(n+1)n}}$$

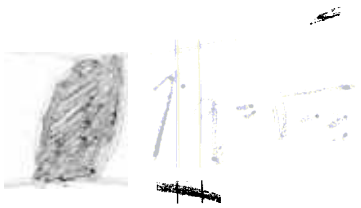
so $s \gg M_D^2$ (transplanckianality) keeps this correction small

Since p $2(n+$ the corrections to the eikonal amplitude g like

$$O\left(\frac{-t}{s}\right) + O\left[\left(\frac{M_D^2}{s}\right)^{\frac{n+2}{n}}\right]$$

small angle

transplanck.an



Cross-section

Cross-section for scattering is

$$\frac{d\sigma_{\text{eik}}}{dt} = \pi b_c^4 \left| F_n(b_c\sqrt{-t}) \right|^2$$

small t.

We are interested in small angle jet-jet scattering at LHC. An observable that signals small angle, which is also z-boost invariant, is rapidity separation $\Delta\eta = \eta_{\text{jet},1} - \eta_{\text{jet},2}$.

$$\frac{-\hat{t}}{\hat{s}} = \frac{1}{1+e^{\Delta\eta}}$$

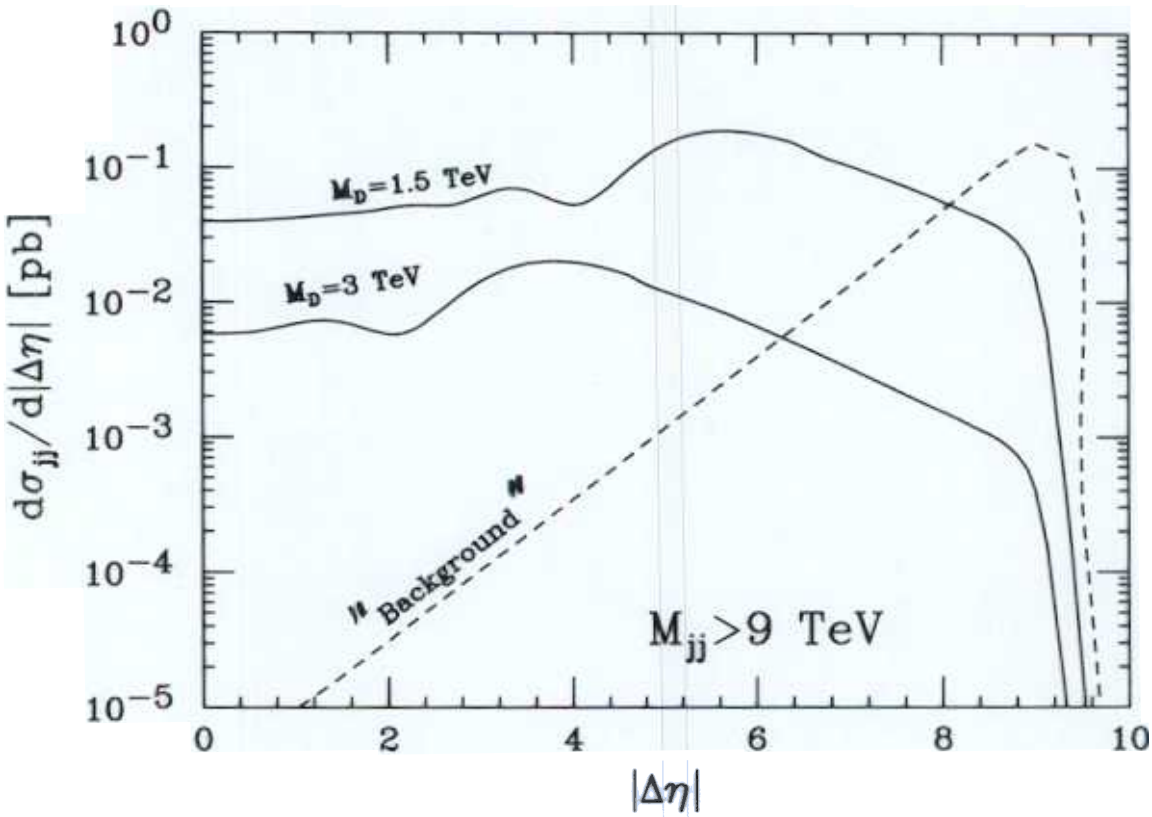
Large rapidity separation implies small $\hat{t}$, consistent with the approximation.

The differential cross-section in $\Delta\eta$ is

$$\frac{d\hat{\sigma}}{d\Delta\eta} = \frac{\pi b_c^4 \hat{s} e^{\Delta\eta}}{(1+e^{\Delta\eta})^2} \left| F_n(y) \right|^2,$$

where $y = b_c\sqrt{\hat{s}}/\sqrt{1+e^{\Delta\eta}}$.

$\Delta\eta$ distribution of jet-jet cross-section

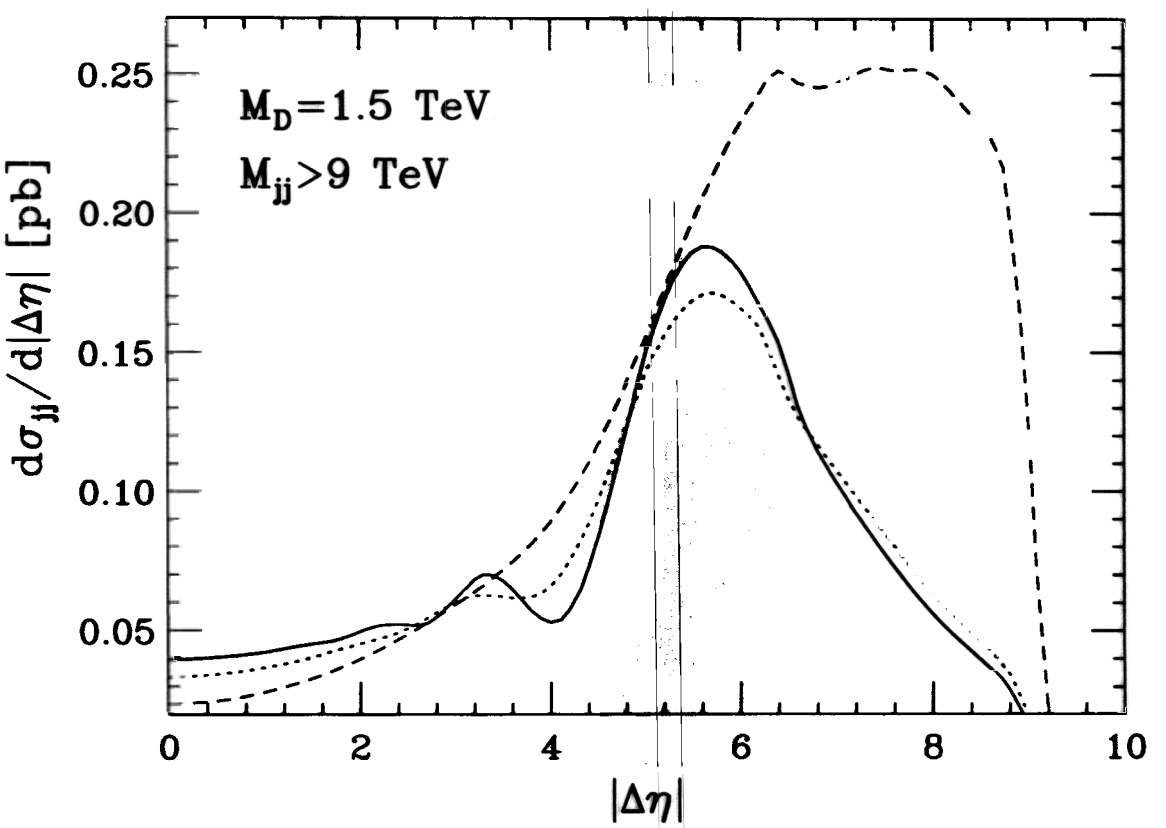


Plot made for $n = 6$, $M_{jj} > 9\,\text{TeV}$, $p_T > 100\,\text{GeV}$, and $|\eta| < 5$. Background from Pythia.

$\Delta\eta$ drops off rapidly at ~ 9 because

$$\Delta\eta \lesssim \ln\left(\frac{\hat{s}}{p_{T,\min}^2}\right)$$

Closer look at wiggles in the distribution



Extrema in the distribution are located at values of $\Delta\eta$ that satisfy the equation

$$1 - e^{-\Delta\eta} - - \frac{y}{|F_n(y)|}\frac{d|F_n(y)|}{dy}\bigg|_{y=\frac{b_c\sqrt{s}}{\sqrt{1+e^{\Delta\eta}}}}$$

First peak at

$$\Delta\eta^{(\text{peak})} \simeq \frac{2(n+2)}{n}\ln\left(\frac{k_n M_{jj}}{M_D}\right)$$

where $k_{2,3,4,5,6}$ 0.8, 0.9, 1.0, 1.2, 1.3.

Wiggles from quantum effects when $q \simeq b_c^{-1}$

Definining a good observable

In practice, experiment will measure distributions of jet-jet cross-section in two variables: $|\Delta\eta|$ and M_{jj}.

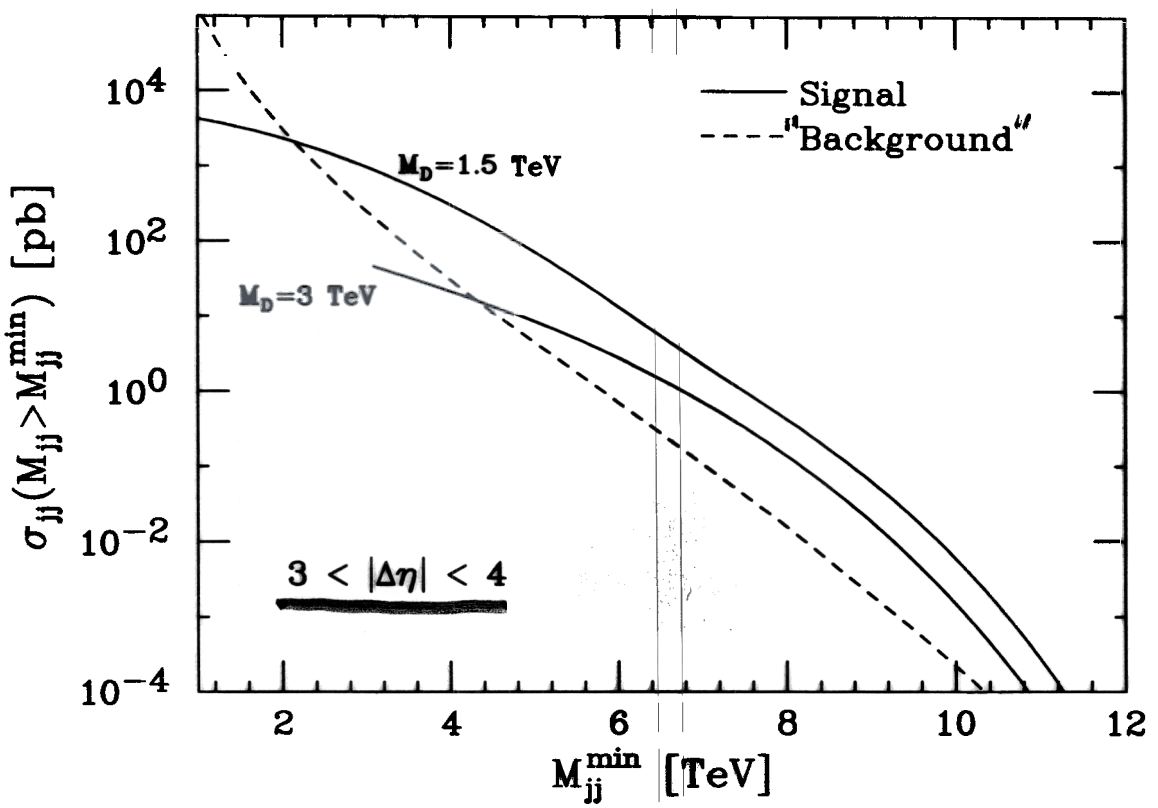
For $\Delta\eta$ we have found
signal $\gg$ background for lower values of $\Delta\eta$
but, signal becomes less calculable at lower $\Delta\eta$.

Choose minimum value of $\Delta\eta$ to be 3, which is equivalent to $q^2/s < 1/20$.

Choose maximum value of $\Delta\eta$ to be 4 for illustration.

Integrating over this range of $\Delta\eta$, we can now show dependence on M_{jj}.

Minimum jet-jet invariant mass



The larger the value of $M_{jj}^{\min}$, the better the signal to background ratio. Gravity signals always rise faster with energy than SM signals.

Good news: Calculability and signal-to-background both like high M_{jj}.