Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTIONS FOR THE FINAL EXAM

Due at 3 PM, December 9th

Guidelines:

- There are two problems in this exam. You are required to work on both problems.
- The only reference you are allowed to use is the textbook. You must work on the exam by yourself. No discussion is allowed with any other person.
- No computer software or calculator is allowed. The only "tool" allowed is pen and paper.
- You are required to derive every equation you write down, except for those derived in the textbook. If you make use of an equation in the textbook, you must cite the original equation in the textbook explicitly. Failure to give the original reference will result in partial or no credit at all.
- It is important for you to clearly state the logic of your answers. I will not make any attempt to "guess" your results. If I cannot follow what you write, I cannot give you the credit.
- Please return your answers to Grant Darktower in the departmental office by the deadline. Or, if you wish, email an electronic version of your answers to me by the deadline.

Problem 1: Non-relativistic Quantum Field Theory

Consider the following Lagrangian with a complex scalar field $\psi(x)$:

$$\mathcal{L}_0 = i\psi^* \partial_t \psi + \alpha \nabla\psi^* \cdot \nabla\psi$$

where b is a real number.

(a) Even though the Lagrangian is not real, show the action $S = \int d^4x\, \mathcal{L}_0$ is. In other words, show that $S = S^*$. Then derive the Euler-Lagrange equations and solve for the plane-wave solutions, for which $\psi = e^{-i(\omega t - \vec{k}\cdot\vec{x})}$. What is the relation between ω and $k \equiv |\vec{k}|$?

(b) The Lagrangian $\mathcal{L}_0$ is not Lorentz invariant. However, it does preserve space-time translations as well as an internal $U(1)$ symmetry. Derive the Noether currents corresponding to the above symmetries. Find the conserved charges associated with the space-time translations and the $U(1)$ symmetry. Express both the current and the charge in terms of the field and its derivatives.

(c) Argue that α must have the dimension of 1/mass. So let's choose a normalization such that $\alpha = \pm 1/(2m)$. Fix the sign of α and explain your reason. Can you literally identify m with the mass?

(d) Canonically quantize the theory by writing down the commutation relations satisfied by ψ and ψ^*, as well as their conjugate momenta. Expand the fields in terms of the plane-wave solution in (a) and identify properly normalized coefficients in the expansion with the creation and/or annihilation operators. Furthermore, write the energy, the linear momentum, and the $U(1)$ charge in terms of the creation/annihilation operators. (Normal-order if need to.)

(e) Find the equation of motion for the two-particle state $|k_1, k_2\rangle$ in the Schroedinger picture. Can you recognize this equation? What is the physical meaning of the $U(1)$ charge? Can you have particle creation and annihilation in this theory?

(f) Now add a Coulomb interaction among the ψ particles (as is often the case for charged particles in condensed matter systerm):

$$\mathcal{L}_0 - \int d^3y\, \psi(\vec{y},t)\psi^*(\vec{y},t)\, \frac{e^2}{|\vec{x}-\vec{y}|}\, \psi(\vec{x},t)\psi^*(\vec{x},t)$$

Compute the scattering amplitude of $\psi(\vec{k}_1)+\psi(\vec{k}_2) \to \psi(\vec{p}_1)+\psi(\vec{p}_2)$ to leading order. Is the interaction attractive or repulsive for this particular process?

Solution:

(a) The action is S is real because the difference in $\mathcal{L}$ and $\mathcal{L}^*$ is a total derivative. The Euler-Lagrange equation for ψ and ψ^* are $i\partial_t\psi = \alpha\nabla^2\psi$ and $i\partial_t\psi^* = -\alpha\nabla^2\psi^*$, respectively. For plane-wave solutions the equation of motion implies $\omega = -\alpha k^2$.

(b) For the space-time translations $x^\mu \to x^\mu - a^\mu$, $\mathcal{L} \to \mathcal{L} + a^\nu(\delta^\mu_\nu\mathcal{L}$, whereas $\delta\psi = a^\nu\partial_\nu\psi$. Then the Noether current is given by

$$(j^\mu)_\nu = \Pi^\mu\partial_\nu\psi - \delta^\mu_\nu\mathcal{L}$$

where $\Pi^\mu \equiv \partial\mathcal{L}/\partial_\mu\psi$ is

$$\Pi^0 = i\psi^*, \quad \Pi^i = -\alpha\partial^i\psi.$$

There are four conserved charges $(j^0)_\nu$, corresponding to the energy $(j^0)_0$ and the three-momentum $P_i = (j^0)_i$. They are

$$H = -\alpha\int d^3x\,\nabla\psi^*\cdot\nabla\psi, \quad P_i = \int d^3x\,i\psi^*\partial_i\psi.$$

For the internal $U(1)$ symmetry, $j^\mu = i\Pi^\mu\psi$ and the charge is

$$N = \int d^3x\psi^*\psi.$$

(c) Since the Lagrangian is first-order in time-derivative, ψ has mass dimension 3/2, which implies the spatial-derivative has mass dimension 5. Therefore α has mass dimension -1. To fix the sign, we look at the energy H derived in (b), which is positive-definite if $\alpha < 0$. So we must choose $\alpha = -1/2m$. In this case the dispersion relation derived in (a) becomes $\omega = +k^2/2m$ which is similar to the dispersion relation in a non-relativistic theory with a particle whose mass is nothing but m.

(d) The canonical momentum of ψ is derived in (b): $\Pi^0 = i\psi^*$ whereas the canonical momentum of ψ^* vanishes. Canonical quantization leads to

$$[\psi(\vec{x},t),\psi^*(\vec{y},t)] = \delta^{(3)}(\vec{x}-\vec{y}), \quad [\psi(\vec{x},t),\psi(\vec{y},t)] = 0, \quad [\psi^*(\vec{x},t),\psi^*(\vec{y},t)] = 0.$$

Now write the plane-wave expansion

$$\psi(\vec{x},t) = \int\frac{d^3k}{(2\pi)^3}\,a_k\,e^{-i(\omega t-\vec{k}\cdot\vec{x})}, \quad \psi^*(\vec{x},t) = \int\frac{d^3k}{(2\pi)^3}\,a^\dagger_k\,e^{i(\omega t-\vec{k}\cdot\vec{x})},$$

It is then straightforward to see that a_k and $a^\dagger_k$ satisfies the commutation relation for the annihilation and creation operator $[a_k, a^\dagger_{k'}] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k'})$.

Using the expressions derived in (b), we find the energy, the linear momentum, and the charge N as follows:

$$\begin{aligned} H &= \int \frac{d^3k}{(2\pi)^3}\,\frac{k^2}{2m}\,a_k^\dagger a_k, \\ P^i &= \int \frac{d^3k}{(2\pi)^3}\,k^i\,a_k^\dagger a_k, \\ N &= \int \frac{d^3k}{(2\pi)^3}\,a_k^\dagger a_k, \end{aligned}$$

where the infinite constants are removed by normal-ordering. Obviously the conserved charge N is the number operator.

(e) The two-particle state is defined as $|k_1, k_2\rangle \equiv a_{k_1}^\dagger a_{k_2}^\dagger |0\rangle$. In the Schroedinger picture the time-evoluation of the state is given by the Schroedinger equation

$$i\frac{\partial}{\partial t}|k_1, k_2\rangle = H|k_1, k_2\rangle = \left(\frac{k_1^2}{2m} + \frac{k_2^2}{2m}\right)|k_1, k_2\rangle.$$

This is just the ordinary Schroedinger equation in non-relativistic quantum mechanics.

In this theory, since the particle number N is a conserved charge, there is no particle creation and annihilation. As is well-known, particle creation and annihilation is a relativistic phenomenon.

(f) Using Dyson's formula, the scattering amplitude at leading order is

$$\begin{aligned} &\langle \psi(p_1)\psi(p_2)|1 - S|\psi(k_1)\psi(k_2)\rangle \\ &= i\mathcal{A}\,(2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2) \\ &= \langle \vec{p_1}, \vec{p_2}|(-i)\int d^4x \int d^3y\,\psi(\vec{y},t)\psi^*(\vec{y},t)\frac{e^2}{|\vec{x}-\vec{y}|}\psi(\vec{x},t)\psi^*(\vec{x},t)|\vec{k}_1, \vec{k}_2\rangle \\ &= 4(-i)\int d^4x \int d^3y\,\frac{e^2}{|\vec{x}-\vec{y}|}e^{-ik_1\cdot x}e^{-ip_1\cdot x}e^{ik_2\cdot y}e^{ip_2\cdot y} \\ &= 4(-ie^2)(2\pi)\int d^3x d^3y\,\frac{e^{i(\vec{k}_1-\vec{p_1})\cdot\vec{x}}e^{i(\vec{k}_2-\vec{p_2})\cdot\vec{y}}}{|\vec{x}-\vec{y}|}\delta(\omega_{k_1} + \omega_{k_2} - \omega_{p_1} - \omega_{p_2}) \end{aligned}$$

where the factor of 4 comes from four different ways of contractions which all result in the same amplitude. Now let's use the relative and CM coordinates:

$$\vec{r} = \vec{x} - \vec{y}, \quad ,\vec{s} = \vec{x} + \vec{y}; \qquad \vec{x} = \frac{1}{2}(\vec{x} + \vec{r}), \quad \vec{y} = \frac{1}{2}(\vec{s} - \vec{r}),$$

so that $d^3xd^y = (1/8)d^3rd^3s$, and the amplitude now becomes

$$
\begin{aligned}
&i\mathcal{A}\,(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2) = \\
&8\pi(-ie^2)\frac{1}{8}\int d^3rd^3s\,\frac{1}{r}e^{\frac{i}{2}(\vec{k_1}-\vec{k_2}-\vec{p_1}+\vec{p_2})\cdot\vec{r}}e^{\frac{i}{2}(\vec{k_1}+\vec{k_2}-\vec{p_2}-\vec{p_2})\cdot\vec{s}}\delta(\omega_{k_1}+\omega_{k_2}-\omega_{p_1}-\omega_{p_2}) \\
&= 4(-ie^2)(2\pi)^4\int d^3r\frac{e^{i\Delta\vec{k}\cdot\vec{r}}}{r}\delta^{(4)}(k_1+k_2-p_1-p_2),
\end{aligned}
$$

where $\Delta\vec{k} = \vec{k_1} - \vec{p_1} = -\vec{k_2} + \vec{p_2}$ is the momentum change of one of the scattered particle. The last step is to perform the Fourier transform of the $1/r$ Coulomb potential. The simpliest solution is to recall the Yukawa potential e^{-mr}/r whose Fourier transform is worked in the textbook to be $4\pi/(|\vec{k}^2+m^2)$. Taking the massless limit gives

$$\int d^3r\frac{e^{i\Delta\vec{k}\cdot\vec{r}}}{r} = \frac{4\pi}{|\Delta\vec{k}|^2}.$$

Alternatively, one could Fourier-transform the Laplace's equation

$$\nabla^2\frac{1}{r} = -4\pi\delta^{(3)}(r)$$

to arrive at the same answer. In the end,

$$i\mathcal{A} = -16i\pi\frac{e^2}{|\Delta\vec{k}|^2}.$$

Comparing with the Yukawa potential in the textbook the amplitude has an extra minus sign and is thus a repulsive interaction.

Problem 2: Neutral Current Interactions at an e^+e^- Collider
In Quantum Electrodynamics the process of electron-positron annihilating into muon pair, mediated by the photon, is the simplest of all QED process and computed in great detail in Peskin and Schroeder. In the early '80s particle physicists discovered a new contribution, in addition to photon, to this process due to the existence of a neutral massive spin-1 particle called the Z boson, which interacts with the electrons and the muons through the following Lagrangian:

$$\mathcal{L}_Z = -\frac{1}{\sqrt{2}}\left(\frac{G_F m_Z^2}{\sqrt{2}}\right)^{\frac{1}{2}}\left[g_R\,\bar{\psi}\gamma^\mu(1+\gamma_5)\psi Z_\mu + g_L\,\bar{\psi}\gamma^\mu(1-\gamma_5)\psi Z_\mu\right],$$

where $\psi = e^-$ or μ^-. In the above $m_Z = 91$ GeV is the mass of the Z and $G_F = 1.66\times10^5\ \mathrm{GeV}^{-2}$ is the Fermi coupling constant. There are now two Feynman diagrams contributing to the annihilation process $e^-(k_1)+e^+(k_2)\to\mu^-(p_1)+\mu^+(p_2)$, as shown in Fig. 1.
(a) Use the following propagator for the Z boson:

$$\frac{-i(g_{\mu\nu}-k_\mu k_\nu/m_Z^2)}{k^2-m_Z^2+i\epsilon}$$

to compute the spin-averaged differential cross-section $d\sigma/dz$ in the centre-of-mass frame where $z\equiv\cos\theta=\vec{k}_1\cdot\vec{p}_1/|\vec{k}_1||\vec{p}_1|$. Express your answer in terms of z and the Mandelstam variable $s=(k_1+k_2)^2$. Also treat the electron and muon as massless since $s>>m_e^2, m_\mu^2$ for practical purposes.

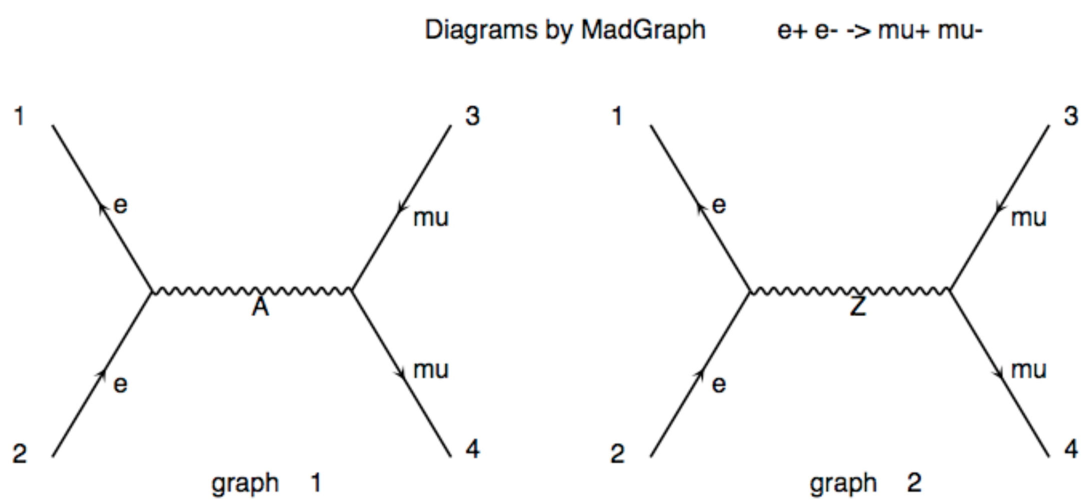

FIG. 1: Feynman diagrams for $e^+e^-\to\mu^+\mu^-$.

(b) There should be three contributions to $d\sigma/dz$: the electromagnetic contribution from the photon γ, the $\gamma - Z^0$ interference, and the effect from the Z^0 diagram alone. At energies much below the Z boson mass, $s << m_Z^2$, determine and justify which one of the three contributions can be neglected. An experimental quantity which allows physicists to observe the effect of the neutral Z boson is the forward-backward asymmetry:

$$A \equiv \frac{\int_0^1 dz \frac{d\sigma}{dz} - \int_{-1}^0 dz \frac{d\sigma}{dz}}{\int_{-1}^1 dz \frac{d\sigma}{dz}}.$$

In the limit $s << m_Z^2$, compute A to leading order in s.

(c) Such an asymmetry was indeed measured to be $-8.1 \pm 2.1\%$ by the MARK-J experiment at DESY in Hamburg, whose centre-of-mass energy is roughly $s \approx 1400$ GeV^2. Using $g_R - g_L = 1$, which is true in the standard model of particle physics, and the fine-structure constant $\alpha \approx 1/137$, estimate the amount of asymmetry predicted by our theory and compare it with the experiment. How many standard deviation away is your prediction from the measured value?

Solution:

(a) The amplitude is

$$\begin{aligned} i\mathcal{A} &= ie^2 \bar{u}(\mu, p_1)\gamma_\lambda v(\mu, p_2)\frac{g^{\lambda\nu}}{s}\bar{v}(e, k_2)\gamma_\nu u(e, k_1) \\ &+ \frac{i}{2}\left(\frac{G_F m_Z^2}{\sqrt{2}}\right)\bar{u}(\mu, p_1)\gamma_\lambda[g_R(1+\gamma_5) + g_L(1-\gamma_5)]v(\mu, p_2) \\ &\times \frac{g^{\lambda\nu}}{s - m_Z^2}\bar{v}(e, k_2)\gamma_\nu[g_R(1+\gamma_5) + g_L(1-\gamma_5)]u(e, k_1), \end{aligned}$$

where we have dropped the longitudinal contribution in the Z boson propagator, which gives zero amplitude upon using Dirac equations. In the CM frame the kinematic constraints are, upon neglecting m_e^2 and m_μ^2,

$$\begin{aligned} k_1 \cdot k_2 &= p_1 \cdot p_2 = \frac{1}{2}s \\ k_1 \cdot p_1 &= k_2 \cdot p_2 = \frac{1}{4}s(1-z) \\ k_2 \cdot p_1 &= k_1 \cdot p_2 = \frac{1}{4}s(1+z). \end{aligned}$$

Then the spin-averaged cross-section is

$$\begin{aligned}\frac{d\sigma}{dz} &= \frac{\pi\alpha^2}{2s}(1+z^2)\\ &\quad +\frac{\alpha G_F m_Z^2(s-m_Z)^2}{8\sqrt{2}(s-m_Z^2)^2}[(g_R+g_L)^2(1+z^2)+2(g_R-g_L)^2 z]\\ &\quad +\frac{G_F^2 m_Z^4 s}{64\pi(s-m_Z^2)^2}[(g_R^2+g_L^2)^2(1+z^2)+2(g_R^2-g_L^2)^2 z],\end{aligned}$$

where $\alpha = e^2/4\pi$ is the fine-structure constant. The first term above is the electromagnetic contribution, the second term the $\gamma - Z^0$ interference, and the third the Z^0 diagram alone.

(b) For $s << m_Z^2$, we see from (a) that the first term goes like α^2/s, the second term like αG_F, and the last term $G_F^2 s$. So clearly we can neglect the last contribution from the Z^0 diagram alone. In this limit,

$$A = -\frac{3G_F s}{16\sqrt{2}\pi\alpha}(g_R - g_L)^2.$$

(c) Given the expression in (b) we see

$$\begin{aligned}A &\approx -6.7\times 10^{-5}\left(\frac{s}{1\ \mathrm{GeV}^2}\right)(g_R-g_L)^2\\ &\rightarrow -10\%(g_R-g_R)^2.\end{aligned}$$

So the amount of asymmetry predicted is roughly -10%, which is only one σ away from the measured value and agrees with experiment quite well!