Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTIONS FOR ASSIGNMENT #10

Reading Assignments:

The subsection on "Mandelstam Variables" in Section 5.4 and the subsection on "Photon Polarization Sums" in Section 5.5 of Peskin and Schroeder.

Problem 1

Do Problem 5.2 in Peskin and Schroeder.

Solution:

There are two diagrams: one from the t-channel exchange and one from the s-channel. Moreover, there is a relative minus sign between the diagram coming from anti-commuting the fermion fields inside the time-ordered product. Using the kinematics $e^-(k_1, r) + e^+(k_2, s) \to e^-(p_1, r') + e^+(p_2, s')$, the amplitude is

$$\begin{aligned} i\mathcal{A} &= i\mathcal{A}_t - i\mathcal{A}_s \\ i\mathcal{A}_t &= +ie^2 \frac{[\bar{u}^{r'}(p_1)\gamma_\mu u^r(k_1)][\bar{v}^{s'}(p_2)\gamma^\mu v^s(k_2)]}{(k_1 - p_1)^2 + i\epsilon} \\ i\mathcal{A}_s &= +ie^2 \frac{[\bar{v}^s(k_2)\gamma_\mu u^r(k_1)][\bar{u}^{r'}(p_1)\gamma^\mu v^{s'}(k_2)]}{(k_1 + k_2)^2 + i\epsilon}. \end{aligned}$$

Therefore there are three terms in the spin-sum from $|\mathcal{A}_t|^2$, $|\mathcal{A}_s|^2$, and the interference between them:

$$\begin{aligned} \frac{1}{4}\sum_{r,s}\sum_{r',s'} |\mathcal{A}|^2 &= \frac{32e^4}{4(k_1 - p_1)^4}\left[(p_1 \cdot p_2)(k_1 \cdot k_2) + (p_2 \cdot k_1)(p_1 \cdot k_2)\right] \\ &\quad + \frac{32e^4}{4(k_1 + k_2)^4}\left[(p_1 \cdot k_1)(p_2 \cdot k_2) + (p_2 \cdot k_1)(p_1 \cdot k_2)\right] \\ &\quad + \frac{32e^4}{4(k_1 - p_1)^2(k_1 + k_2)^2}\left[(k_1 \cdot p_2)(p_1 \cdot k_2) + (p_2 \cdot k_1)(p_1 \cdot k_2)\right]. \end{aligned}$$

The kinematics, on the other hand, can be written as follow when ignoring the electron mass:

$$\begin{aligned} s &= 2k_1 \cdot k_2 = 2p_1 \cdot p_2 \\ t &= -2p_1 \cdot k_1 = -2p_2 \cdot k_2 \\ u &= -2p_1 \cdot k_2 = -2p_2 \cdot k_1. \end{aligned}$$

Now the amplitude-squared becomes

$$\frac{1}{4}\sum_{r,s}\sum_{r',s'} |\mathcal{A}|^2 = 2e^4 \left(\frac{u^2 + s^2}{t^2} + \frac{t^2 + s^2}{u^2} + \frac{2u^2}{st} \right).$$

The differential cross-section is

$$\frac{d\sigma}{d(\cos\theta)} = \frac{2\pi|\mathcal{A}|^2}{64\pi^2 E_{cm}^2} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right].$$

The scattering angle in the CM frame is $s = 4E^2$, $t = -2E^2(1-\cos\theta)$, and $u = -2E^2(1+\cos\theta)$. So the t-channel propagator is causing the cross-section to diverge in $\theta \to 0$.

Problem 2

Consider the following Lagrangian for the *scalar* Quantum Electrodynamics:

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - m_\phi^2|\phi|^2 = (\partial_\mu + ieA_\mu)\phi^*\,(\partial^\mu - ieA^\mu)\phi - m_\phi^2\phi^*\phi.$$

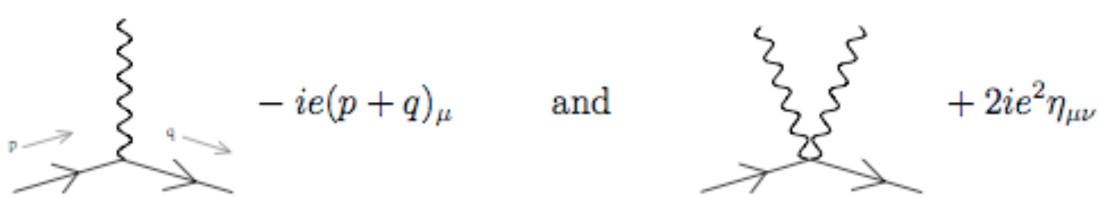
(a) Write down the Feynman rule for the interaction vertices in this theory. Be careful to explain and take into account symmetry factors, if any. (There is actually a subtlety involving the one-derivative term in the interactions, but you can get the correct answer by deriving the Feynman rules in the *naive* way.)
(b) Compute the amplitude for the scattering process $\phi(k_1) + \phi^*(k_2) \to \phi(p_1) + \phi^*(p_2)$ to the lowest order in e using the following massless propagator for the photon:

$$\frac{-ig_{\mu\nu}}{k^2 + i\epsilon}.$$

(c) Compute the amplitude for the scattering process $\phi(k_1) + \gamma(k_2) \to \phi(p_1) + \gamma(p_2)$ to the lowest order in e using Feynman rules/diagrams. Be sure to include all possible diagrams.
(d) Now assume the photon is massive and has the following propagator

$$\frac{-i\left(g_{\mu\nu} - k_\mu k_\nu/m_A^2\right)}{k^2 - m_A^2 + i\epsilon}.$$

Repeat (b) and show that the amplitude has a smooth limit in taking $m_A \to 0$. (In other words, the $k_\mu k_\nu/m_A^2$ piece in the massive propagator doesn't contribute to the amplitude, just like in the ordinary QED with fermions.)
(e) Write the amplitudes in (c) as $\epsilon_\mu^{(i)}\epsilon_\nu^{(f)}\mathcal{M}^{\mu\nu}$. Show that $k_2^\mu\mathcal{M}_{\mu\nu} = p_2^\nu\mathcal{M}_{\mu\nu} = 0$, again like in the ordinary QED with fermions.
(f) Show that the Coulomb potential resulting from the process in (b) is attractive, whereas it is repulsive for the $\phi + \phi \to \phi + \phi$ scattering.
Solution:
(a) See the Figure. The factor of two in the "seagull diagram" is because there are two identical particles in the vertex.



(b) There are two contributions from the s-channel and t-channel diagrams, respectively,

$$i\mathcal{A} = +ie^2 \left[(k_1 - k_2)^\mu \frac{g_{\mu\nu}}{(k_1+k_2)^2 + i\epsilon} (p_1 - p_2)^\nu + (k_1 + p_1)^\mu \frac{g_{\mu\nu}}{(k_2 - p_1)^2 + i\epsilon} (-k_2 - p_2)^\nu \right]$$

The extra minus sign in the numerator in the t-channel contribution comes from the fact that the momentum flow in the Feynman rule is correlated with the particle number flow in the diagram.
(c) There are three diagrams, two from the left vertex in the figure (with the two external photons exchanged), and one from the right vertex.

$$i\mathcal{A} = -ie^2 \epsilon(k_2)_\mu \epsilon(p_2)^*_\nu \left[\frac{(2k_1+k_2)^\mu (2p_1+p_2)^\nu}{(k_1+k_2)^2 + i\epsilon} + \frac{(2p_1-k_2)^\mu (2k_1-k_2)^\nu}{(p_1-k_2)^2 + i\epsilon} - 2g^{\mu\nu} \right].$$

(d) For the s-channel contribution

$$i\mathcal{A} = +ie^2 (k_1 - k_2)^\mu \frac{(g_{\mu\nu} - k_\mu k_\nu / m_A^2)}{k^2 - m_A^2 + i\epsilon} (p_1 - p_2)^\nu.$$

Since $k = k_1 + k_2 = p_1 + p_2$, it is immediately clear that the longitudinal piece gives zero contribution. One can show easily that this is the case for the t-channel contribution as well. This can be related to the current conservation in the scalar QED:

$$j^\mu = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*), \quad \partial_\mu j^\mu = 0.$$

(e) Given

$$\mathcal{M}^{\mu\nu} = \frac{(2k_1+k_2)^\mu (2p_1+p_2)^\nu}{(k_1+k_2)^2 + i\epsilon} + \frac{(2p_1-k_2)^\mu (2k_1-k_2)^\nu}{(p_1-k_2)^2 + i\epsilon} - 2g^{\mu\nu}.$$

It is simple to check that $k_2^\mu \mathcal{M}_{\mu\nu} = p_2^\nu \mathcal{M}_{\mu\nu} = 0$.
(f) In the non-relativistic limit the leading contribution is from only the t-channel diagram and the numerator in the amplitude is $(-k_2 - p_2)^\mu (k_1 + p_1)_\mu \approx (-k_2 - p_2)^0 (k_1 + p_1)_0 \sim -(2m_\phi)^2$. So the amplitude is

$$i\mathcal{A} \approx +ie^2 \frac{4m_\phi^2}{|\vec{p_1} - \vec{k_2}|^2}$$

which has the same sign as the Yukawa potential and is thus attractive. On the other hand, for the $\phi + \phi \to \phi + \phi$ scattering there is no extra minus sign in the t-channel contribution due to the momentum flow (as explained in (b)), so the potential is repulsive.