Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTIONS FOR ASSIGNMENT #1

Reading Assignments:
(a) Read the pages on "Notations and Conventions" in Peskin and Schroeder.
(b) F. Wilczek, "Quantum field theory," Rev. Mod. Phys. **71**, S85 (1999), which can be downloaded at `http://arxiv.org/abs/hep-th/9803075`
It is all right if you don't understand everything he says, but you should try to read it over and over again until you think you have understood "something!"

Problem 1
Explain what is the phenomenon of "dimensional transmutation?"
(Hint: if you have done your reading assignment, you will be able to answer it.)
Solution:
See Section III of Wilczek's article.

Problem 2
Here are a few exercises on dimensional analysis and the Natural Unit system $c = \hbar = 1$.
(a) The proton mass in the SI unit is $m_p = 1.62 \times 10^{-27}$ kg. Convert m_p into the Natural Unit. The Large Hadron Collider (LHC) at CERN in Geneva is a proton-proton collider designed to have a center-of-mass energy of 14 TeV. What is the speed of the proton, expressed in terms of the speed of light c, when the LHC is operating at the designed CM energy?
(b) In SI unit Maxwell's equations contain three dimensionful coupling constants: the electric charge of the electron $e = 1.62 \times 10^{-19}$ C, the permittivity of free space $\epsilon_0 = 8.854 \times 10^{-12}$ F/m, and the permeability of free space $\mu_0 = 1/(\epsilon_0 c^2)$ which can be traded in for the speed of light c. Can you generate a quantity out of e, ϵ_0, c, and $\hbar$ that is *dimensionless*? Is there more than one dimensionless quantity that can be generated?
(c) Divide your answer(s) in (b) by 4π and call it α. What is the numerical value of α? Approximate α by a fraction $\alpha \approx 1/N$ where N is an integer. What is N? Can you recognize that α is a well-known fundamental constant?
(d) The Newton's constant $G_N = 6.674 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}. In the Natural Unit it has the mass dimension of -2, which is used to define a Planck mass $M_p = 1.22 \times 10^{-19}$ GeV. Convert M_p into the SI unit by expressing it in terms of G_N, $\hbar$, and c.
(e) With a CM energy of 14 TeV, typical energy scales at the LHC will be at around TeV. For a typical quantum mechanical amplitude at the LHC, how large is the correction to the amplitude due to effects of gravity?
Solution:
(a) $m_p = 938$ MeV ≈ 1 GeV. At 14 TeV CM energy, each proton has an energy of 7 TeV = 7000 GeV. The speed of the proton v must be so that

$$7000 = \frac{1}{\sqrt{1 - v^2}},$$

which implies $v = 1 - 10^{-8} = 0.99999999$ times the speed of light c.
(b) $e^2/(\epsilon_0 \hbar c)$ is the only dimensionless constant.
(c) $\alpha = e^2/(4\pi\epsilon_0 \hbar c) \approx 1/137$ is the fine-structure constant.
(d) $M_p = (\hbar c^5)/G_N$.
(e) The gravitational correction is controlled by the Newton's constant G_N, so the correction must go like $G_N E^2 = (E/M_p)^2$, which is the only dimensionless number made out of the scales E and G_N. For $E \approx 10^3$ GeV, this correction is roughly 10^{-26}!!

Problem 3
During the class we showed that the following amplitude

$$\langle \vec{x} | e^{iHt} | \vec{x} = 0 \rangle = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot(x-y)} \tag{1}$$

is non-vanishing outside of the forward light cone of the particle. However, you may protest that the integration measure d^3k is not invariant under Lorentz transformation and that is why causality is violated. One could, instead, use a relativistically invariant amplitude as follows

$$D(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} (2\pi) \delta(k^2 - m^2) \Big|_{k^0 > 0} e^{-ik\cdot(x-y)}. \tag{2}$$

Use the technique we used in class to show that even this Lorentz-invariant amplitude violates causality in that it still is non-vanishing outside of the forward light cone. There is actually a shortcut to this problem that you are *not* allowed to use:

$$\frac{\partial}{\partial t} D(x-y) = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot(x-y)}. \tag{3}$$

However, you can use the above relation to check your answer.
Solution:
Use the identity

$$\delta(k^2 - m^2) = \frac{1}{2\omega_k} \left(\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k) \right), \tag{4}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$,

$$D(x-y) = \int \frac{d^3k}{2\omega_k} \frac{1}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} e^{-i\omega_k \cdot t} \tag{5}$$

where $t = x^0 - y^0$. After performing the straightforward angular integration, like we did in class, and defining $r = |\vec{x} - \vec{y}|$,

$$D(x-y) = -\frac{i}{2\pi^2 r} \int_{-\infty}^{\infty} \frac{|\vec{k}|}{\omega_k} d|\vec{k}| e^{i|\vec{k}|r} e^{-i\omega_k t}. \tag{6}$$

As explained in class, the integrand is analytic everywhere in the complex plane except for the branch cuts along the imaginary axis starting at $|\vec{k}| = \pm im$. We are interested in the case when $r \gg t$. The integrand then vanishes at infinity in the upper plane and the integral is equivalent to an integral along the upper branch cut:

$$D(x-y) = -\frac{i}{2\pi^2 r}\int_m^\infty \frac{i\ell}{\sqrt{\ell^2-m^2}} d(i\ell) e^{-\ell r}\left(e^{\sqrt{\ell^2-m^2}t} + e^{-\sqrt{\ell^2-m^2}t}\right) \quad (7)$$

$$= \frac{i}{\pi^2 r} e^{-mr}\int_m^\infty d\ell\ \ell e^{-(\ell-m)r}\frac{\cosh\left(\sqrt{\ell^2-m^2}\right)}{\sqrt{\ell^2-m^2}}. \quad (8)$$

The main difference between $D(x-y)$ and its non-relativistic counter part is the extra $1/\omega_k$ factor in $D(x-y)$. Since ω_k changes sign when one goes across the branch cut, the integrand is $e^{\mathrm{Im}\,\omega_k t}/\mathrm{Im}\,\omega_k$ along one side of the cut and $e^{-\mathrm{Im}\,\omega_k t}/(-\mathrm{Im}\,\omega_k)$ along the other side of the cut. This results in the relative plus sign in the integrand for $D(x-y)$ whereas it is a relative minus sign for the non-relativistic counter part.

One can see that $D(x-y)$ is non-vanishing outside of the forward light cone since the integrand is positive. Moreover, it is easy to see that indeed Eq. (3) is true.