Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html

SOLUTIONS FOR ASSIGNMENT #2

Reading Assignments:
(a) Read the handout by Professor Michael Dine at UC Santa Cruz on relativity at http://scipp.ucsc.edu/~dine/ph217/217relativity.pdf
(b) Section 3.1 of Peskin and Schroeder.

Problem 1
Do exercises a, e, and h in the reading assignment (a).
Solution:
(a)

$$\Lambda = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix}$$

(e) In the Lab frame the proton is at rest so $k_p = (m_p, \vec{0})$ and $k_e = (\omega_e, \vec{k}_e)$. Then $s = (k_e + k_p)^2 = (m_p + \omega_e)^2 - |\vec{k}_e|^2$.
(h) The tensor transforms as

$$F^{\mu\nu\,\prime} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \, F^{\alpha\beta},$$

where Λ^μ_α is given in (a). The electric field is $E^i = F^{i0}$ and the magnetic field is $B^i = (1/2)\epsilon^{i0\mu\nu}F_{\mu\nu}$. Therefore we have

$$E^{i\,\prime} = \Lambda^i_\alpha \Lambda^0_\beta \, F^{\alpha\beta}$$

$$B^{i\,\prime} = (1/2)\epsilon^{i0\mu\nu} F'_{\mu\nu} = (1/2)\epsilon^{i0\mu\nu}\, \Lambda^\alpha_\mu \Lambda^\beta_\nu \, F_{\alpha\beta}$$

Plug in Λ to get the explicit expression.

Problem 2
Do Problem 3.1 in Peskin and Schroeder. (We haven't introduced a Dirac spinor yet. However, you have all learned a two-component Pauli spinor in Quantum Mechanics. For the purpose of this problem, think of the two-component Weyl spinor as the Pauli spinor, which should enable you to do the problem.)
Solution:
(a)

$$[L^i, L^j] = i\epsilon^{ijk} L^k, \quad [L^i, K^j] = i\epsilon^{ijk} K^k, \quad [K^i, K^j] = -i\epsilon^{ijk} L^k.$$

From the above it is simple to show that

$$[J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm, \quad [J^i_+, J^j_-] = 0$$

(b) Using the relation

$$L^j = (J^j_+ + J^j_-), \quad K^j = -i(J^j_+ - J^j_-)$$

we see that an infinitesimal Lorentz transformation can now be written as

$$\Phi \to \left(1 - \frac{1}{2}J^i_+(i\theta^i + \beta^i) - \frac{1}{2}J^i_-(i\theta^i - \beta^i)\right)\Phi.$$

For the $(\frac{1}{2}, 0)$ representation, set $J^i_+ = \sigma^i/2$ and $J^i_- = 0$, whereas for the $(0, \frac{1}{2})$ representation set $J^i_- = \sigma^i/2$ and $J^i_+ = 0$. Then we see that these are exactly the transformation law of ψ_L and ψ_R given in (3.37) of Peskin and Schroeder.

(c) Define $\bar{\sigma}_\mu = (1, \vec{\sigma})$ and write

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^\mu \bar{\sigma}_\mu,$$

which now transforms as

$$\begin{aligned} V^\mu \bar{\sigma}_\mu &\to \left(1 - \frac{\sigma^i}{2}(i\theta^i - \beta^i)\right) V^\mu \bar{\sigma}_\mu \left(1 + \frac{\sigma^i}{2}(i\theta^i + \beta^i)\right) \\ &\to V^\mu \bar{\sigma}_\mu - \frac{\sigma^i}{2}(i\theta^i - \beta^i)\, V^\mu \bar{\sigma}_\mu + V^\mu \bar{\sigma}_\mu \, \frac{\sigma^i}{2}(i\theta^i + \beta^i) + \mathcal{O}(\beta^2, \theta^2) \\ &= V^\mu \bar{\sigma}_\mu + \frac{1}{2} V^\mu \beta^i \{\sigma^i, \bar{\sigma}_\mu\} - \frac{i}{2} V^\mu \theta^i [\sigma^i, \bar{\sigma}_\mu] \end{aligned}$$

Now use the following identities

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

one can compute

$$\begin{aligned} V^\mu \bar{\sigma}_\mu &\to V^\mu \bar{\sigma}_\mu + V^0 \beta^i \sigma^i + V^i \beta^i + V^j \theta^i \epsilon^{ijk} \sigma^k \\ &= (\delta^\mu{}_\nu + \omega^\mu{}_\nu) V^\nu \bar{\sigma}_\mu \end{aligned}$$

where $\omega^i{}_0 = \omega^0{}_i = \beta^i$ and $\omega^j{}_i = -\omega^i{}_j = \epsilon^{ijk}\theta^k$. Using $\omega_{\mu\nu} = g_{\mu\alpha}\omega^\alpha{}_\nu$ it is clear that $\omega_{\mu\nu}$ is anti-symmetric. Therefore V^μ transforms like a Lorentz vector as in Eq. (3.19) in Peskin and Schroeder.

Problem 3

We would like to consider a two-dimensional quantum harmonic oscillator with the following Hamiltonian in Cartesian coordinate:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

where I have set $m = \omega = 1$ for simplicity. The operators satisfy $[x_i, p_j] = -i\delta_{ij}$

(a) Write down the Hamiltonian in terms of creation and annihilation operators $\{a_1^\dagger, a_2^\dagger, a_1, a_2\}$, which satisfy the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$. (All other commutators are zero.)

(b) The Hamiltonian obviously has rotational invariance in the two-dimensional space:

$U(R)x_iU(R)^\dagger = D(R)_{ij}x_j$, where $D(R)$ is a 2×2 orthogonal matrix. A less obvious invariance is a complex rotation S in (a_1, a_2),

$$U(S)\,a_i\,U(S)^\dagger = D(S)_{ij}a_j,$$

where $D(S)$ is a complex 2×2 matrix. Work out the condition on $D(S)$ in order for the Hamiltonian to be invariant under $S : U(S)\,H\,U(S)^\dagger = H$. Show that different states related by an S transformation $|a\rangle = U(S)|b\rangle$ are degenerate in energy.

(c) Define the one-particle states $\{|i\rangle = a_i^\dagger|0\rangle, i = 1, 2\}$. We discussed in class that any operator can be expressed in terms of creation and annihilation operators. Consider a set of operators $\{T^a, a = 1, 2, 3\}$ whose effects on the one-particle states are

$$T^a|0\rangle = 0, \qquad T^a|i\rangle = \frac{1}{2}|j\rangle[\sigma^a]_{ij}$$

where σ^a's are the Pauli matrices and $[\sigma^a]_{ij}$ are the matrix elements of Pauli matrices. Find the representation of T^a in terms of the creation and annihilation operators.

(d) Use your result in (c) to compute the commutators $[T^a, a_i^\dagger]$.

<u>Solution</u>:

(a)

$$H = (a_1^\dagger\ a_2^\dagger)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) From (a) we see that the matrix $D(S)$ must satisfy $D(S)^\dagger D(S) = 1$. Since $[H, U(S)] = 0$, $H(U(S)|b\rangle) = U(S)(H|b\rangle) = E_b(U(S)|b\rangle)$ so $U(S)|b\rangle$ and $|b\rangle$ have the same energy.

(c) T^a maps a one-particle state into another one-particle state, so it must be the product of one creation and one annihilation operator. Use the notation $\alpha = (a_1\ a_2)^T$, it is straightforward to work out that

$$T^a = \frac{1}{2}\alpha^\dagger\sigma^a\alpha$$

where σ^a are the Pauli matrices.

(d)

$$[T^a, a_i^\dagger] = \frac{1}{2}a_j^\dagger[\sigma^a]_{ji}$$

where $[\sigma^a]_{ij}$ is the matrix element of σ^a.

A physical system corresponding to such a Hamiltonian is that of a proton and a neutron. The extra symmetry exhibited here is called the “isospin symmetry.”