Northwestern University Department of Physics and Astronomy

# PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

*ASSIGNMENT #3*

Due at 3:30 PM, October 16th

(Two pages and three problems.)

**Reading Assignments:**

(a) Read Section 15.4 of Peskin and Schroder on Lie algebras.

(b) Read Chapter 2 of Peskin and Schroeder.

**Problem 1**

(a) Write down the generator of $SU(2)$ in the adjoint representation.

(b) Do Problem 15.1 in Peskin and Schroeder.

**Problem 2**

Consider a free spinless boson $\phi(x)$ with the following plane-wave expansion:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (a_k\, e^{-ik\cdot x} + a_k^\dagger\, e^{ik\cdot x}).$$

(a) Suppose we treat $\phi(x)$ as a quantum field and canonically quantize it using the commutation relations:

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = [\partial_t\phi(\vec{x},t), \partial_t\phi(\vec{y},t)] = 0, \quad [\phi(\vec{x},t), \partial_t\phi(\vec{y},t)] = i\delta^{(3)}(x-y).$$

Express $a_k$ and $a_k^\dagger$ in terms of $\phi(\vec{x},0)$ and $\partial_t\phi(\vec{x},0)$ and show they satisfy the commutation relations for creation and annihilation operators: $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$ and $[a_k, a_{k'}^\dagger] = \delta^{(3)}(k-k')$.

(b) Alternatively, treat $\phi(x)$ as the simplest quantum field constructed out of the creation and annihilation operators $a_k$ and $a_k^\dagger$ and show that $\phi(x)$ and $\partial_t\phi(x)$ satisfy the correct commutation relations as required by the canonical quantization.

(c) In canonical quantization the Hamiltonian of a free spinless boson can be written as

$$H = \int d^3x \frac{1}{2}\left[(\partial_t\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right].$$

Verify explicitly that

$$H = \int d^3x\ \omega_k\ \left(a_k a_k^\dagger + \frac{1}{2}\delta^{(3)}(0)\right).$$

**Problem 3**

The infinite constant in the Hamiltionian in Problem 2 (c),

$$H_{CC} = \int d^3x\ \omega_k\ \frac{1}{2}\delta^{(3)}(0),$$

actually contains two types of infinities:
(a) The infinity in $\delta^{(3)}(0)$ comes about because the space in which our QFT lives is infinite in volume. To see this explicitly, recall that $\delta^{(3)}(0)$ arises from the commutator

$$[a_k, a_{k'}^\dagger] = \delta^{(3)}(k - k') = \int d^3x \; e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

Use the above equation to show that if we had placed the QFT in a box with sides of length $L$, then $[a_k, a_k^\dagger] = L^3$ which is the volume of the box. Now take the length $L \to \infty$ and show that

$$H_{CC} = \int d^3x \; \frac{1}{2}\omega_k \; V$$

where $V$ is the volume of the infinite space. An infinity associated with an infinite volume is called the *infrared divergence*.
(b) The infrared divergence comes about because we are computing the *total* energy of the system. We could instead compute the energy density $\mathcal{H}_{CC} \equiv H_{CC}/V$ to get around the infrared divergence. Show that there is still a divergence in $\mathcal{H}_{CC}$ because we assume the QFT is valid up to arbitrarily high energy and therefore integrate over arbitrarily high momentum $|\vec{k}|$. Such a divergence is called the *ultraviolate divergence*.
(c) Since no one knows how to write down a consistent QFT for gravity, it is reasonable to assume that our QFT is valid only up to the Planck energy $M_{pl}$ when the effect of gravity becomes important. Therefore we should cut off the $|\vec{k}|$ integral at $M_{pl}$. Calculate the zero-point energy density $\mathcal{H}_{CC}$ in terms of $M_{pl}$.
(d) One way to remove the zero-point energy is to add a so-called *cosmological constant* term to the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \Lambda_{CC}.$$

Show that the total zero-point energy density now becomes

$$\mathcal{E}_{\text{total}} = \mathcal{H}_{CC} - \Lambda_{CC}$$

(e) Over the last decade our colleagues in cosmology worked very hard and measured $\mathcal{E}_{\text{total}} \approx (10^{-3}\text{ eV})^4$ in our universe. Assuming that QFT is indeed only valid up to $M_{pl}$, what is the amount of cancellation needed between $\mathcal{H}_{CC}$ and $\Lambda_{CC}$ in order to result in the observed value? One measure of the *fine-tuning* necessary is to estimate the order of magnitude of

$$\frac{\mathcal{H}_{CC} - \Lambda_{CC}}{\mathcal{H}_{CC} + \Lambda_{CC}}.$$

This is the famous cosmological constant problem!