Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTIONS FOR ASSIGNMENT #3

Reading Assignments:
(a) Read Section 15.4 of Peskin and Schroder on Lie algebras.
(b) Read Chapter 2 of Peskin and Schroeder.

Problem 1
(a) Write down the generator of $SU(2)$ in the adjoint representation.
(b) Do Problem 15.1 in Peskin and Schroeder.
(c) Find four different SU(2) subalgebras of the SU(3).
Solution:
(a) Using $(t^b_G)_{ac} = if^{abc} = i\epsilon^{abc}$ for $SU(2)$ we have

$$t^1_G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^2_G = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad t^3_G = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b)
(b-a) In this basis the $SU(3)$ generators are 3×3 traceless Hermitian matrices. Hermiticity put $3^2 = 9$ constraints whereas the tracelessness adds an additioinal constraint. So in the end there are only $2 \times 3^2 - 3^2 - 1 = 8$ generators.
(b-b) $f^{146} = 0, f^{257} = 1/2, f^{458} = \sqrt{3}/2$.
(b-c) $C(N) = 1/2$.
(b-d) $C_2(N) = 4/3$.
(c) It is clear that $\{t^1, t^2, t^3\}$, $\{t^4, t^5, t^3/2+\sqrt{3}t^8/2\}$, and $\{t^6, t^7, -t^3/2+\sqrt{3}t^8/2\}$ each forms an $SU(2)$ subalgebra. For the fourth subalgebra one needs to look at the structure constants and see which three generators close into themselves. There are many possibilities. One example is $\{2t^2, 2t^5, 2t^7\}$, which is nothing but the adjoint representation of the $SU(2)$ algebra written down in (a).

Problem 2
Consider a free spinless boson $\phi(x)$ with the following plane-wave expansion:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} (a_k^\dagger\, e^{-ik\cdot x} + a_k\, e^{ik\cdot x}).$$

(a) Suppose we treat $\phi(x)$ as a quantum field and canonically quantize it using the commutation relations:

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = [\partial_t\phi(\vec{x}, t), \partial_t\phi(\vec{y}, t)] = 0, \quad [\phi(\vec{x}, t), \partial_t\phi(\vec{y}, t)] = i\delta^{(3)}(x - y).$$

Express a_k and $a_k^\dagger$ in terms of $\phi(\vec{x},0)$ and $\partial_t\phi(\vec{x},0)$ and show they satisfy the commutation relations for creation and annihilation operators: $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$ and $[a_k, a_{k'}^\dagger] = \delta^{(3)}(k-k')$.
(b) Alternatively, treat $\phi(x)$ as the simplest quantum field constructed out of the creation and annihilation operators a_k and $a_k^\dagger$ and show that $\phi(x)$ and $\partial_t\phi(x)$ satisfy the correct commutation relations as required by the canonical quantization.
(c) In canonical quantization the Hamiltonian of a free spinless boson can be written as

$$H = \int d^3x \frac{1}{2}\left[(\partial_t\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right].$$

Verify explicitly that

$$H = \int d^3x\ \omega_k\ \left(a_k a_k^\dagger + \frac{1}{2}\delta^{(3)}(0)\right).$$

Solution:
(a) From

$$\phi(\vec{x},0) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(a_k^\dagger\, e^{i\vec{k}\cdot\vec{x}} + a_k\, e^{-i\vec{k}\cdot\vec{x}}),$$
$$\partial_t\phi(\vec{x},0) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_k}{2}}(a_k^\dagger\, e^{i\vec{k}\cdot\vec{x}} + a_k\, e^{-i\vec{k}\cdot\vec{x}}),$$

one could use the inverse Fourier transform and the Fourier representation of the delta function

$$\int d^3x\ e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta^{(3)}(\vec{k}-\vec{k}')$$

to derive

$$a_k^\dagger = \int d^3x\sqrt{\frac{\omega_k}{2}}\left[\phi(\vec{x},0) - \frac{i}{\omega_k}\partial_t\phi(\vec{x},0)\right]e^{-i\vec{k}\cdot\vec{x}}$$
$$a_k = \int d^3x\sqrt{\frac{\omega_k}{2}}\left[\phi(\vec{x},0) + \frac{i}{\omega_k}\partial_t\phi(\vec{x},0)\right]e^{-i\vec{k}\cdot\vec{x}}$$

. From here it is straightforward to compute the commutation relations and show that

$$[a_k, a_{k'}^\dagger] = \delta^{(3)}(\vec{k}-\vec{k}'), \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0.$$

(b) Using the first two equations in (a) as well as the commutation relations for a_k and $a_k^\dagger$ it is straightforward to prove that the canonical quantization follows.
(c) Again simple and straightforward algebraic manipulations.

Problem 3
The infinite constant in the Hamiltionian in Problem 2 (c),

$$H_{CC} = \int d^3x\ \omega_k\ \frac{1}{2}\delta^{(3)}(0),$$

actually contains two types of infinities:
(a) The infinity in $\delta^{(3)}(0)$ comes about because the space in which our QFT lives is infinite in volume. To see this explicitly, recall that $\delta^{(3)}(0)$ arises from the commutator

$$[a_k, a_{k'}^\dagger] = \delta^{(3)}(k-k') = \int d^3x\ e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

Use the above equation to show that if we had placed the QFT in a box with sides of length L, then $[a_k, a_k^\dagger] = L^3$ which is the volume of the box. Now take the length $L \to \infty$ and show that

$$H_{CC} = \int d^3x \, \frac{1}{2}\omega_k \, V$$

where V is the volume of the infinite space. An infinity associated with an infinite volume is called the *infrared divergence*.

(b) The infrared divergence comes about because we are computing the *total* energy of the system. We could instead compute the energy density $\mathcal{H}_{CC} \equiv H_{CC}/V$ to get around the infrared divergence. Show that there is still a divergence in $\mathcal{H}_{CC}$ because we assume the QFT is valid up to arbitrarily high energy and therefore integrate over arbitrarily high momentum $|\vec{k}|$. Such a divergence is called the *ultraviolate divergence*.

(c) Since no one knows how to write down a consistent QFT for gravity, it is reasonable to assume that our QFT is valid only up to the Planck energy M_{pl} when the effect of gravity becomes important. Therefore we should cut off the $|\vec{k}|$ integral at M_{pl}. Calculate the zero-point energy density $\mathcal{H}_{CC}$ in terms of M_{pl}.

(d) One way to remove the zero-point energy is to add a so-called *cosmological constant* term to the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \Lambda_{CC}.$$

Show that the total zero-point energy density now becomes

$$\mathcal{E}_{\text{total}} = \mathcal{H}_{CC} - \Lambda_{CC}$$

(e) Over the last decade our colleagues in cosmology worked very hard and measured $\mathcal{E}_{\text{total}} \approx (10^{-3}\text{ eV})^4$ in our universe. Assuming that QFT is indeed only valid up to M_{pl}, what is the amount of cancellation needed between $\mathcal{H}_{CC}$ and Λ_{CC} in order to result in the observed value? One measure of the *fine-tuning* necessary is to estimate the order of magnitude of

$$\frac{\mathcal{H}_{CC} - \Lambda_{CC}}{\mathcal{H}_{CC} + \Lambda_{CC}}.$$

This is the famous cosmological constant problem!

Solutions:

(a) It is easy to see that

$$[a_k, a_k^\dagger] = \int_{\text{Box}} d^3x = L^3 = V.$$

Therefore

$$H_{CC} = \frac{1}{2}\int d^3k \; \omega_k [a_k, a_k^\dagger] = \frac{1}{2}\int d^3k \; \omega_k \; V.$$

(b)

$$\mathcal{H}_{CC} = H_{CC}/V = \frac{1}{2}\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\theta \int_0^\infty |\vec{k}|^2 d|\vec{k}| \sqrt{|\vec{k}|^2 + m^2}$$

is still divergent because of the infinite $d|\vec{k}|$ integral.

(c) We can approximate the integral by dropping the mass term since M_{pl} is so large:

$$\mathcal{H}_{CC} = 2\pi \int_0^{M_{pl}} |\vec{k}|^2 d|\vec{k}| \sqrt{|\vec{k}|^2 + m^2} \approx 2\pi \int_0^{M_{pl}} |\vec{k}|^3 d|\vec{k}| = \frac{\pi}{2} M_{pl}^4$$

(d) With the cosmological constant term the Hamiltonian density becomes

$$\mathcal{H} = \frac{1}{2}\left[(\partial_t\phi)^2 + (\nabla\phi)^2 + m^2\phi^2 - \Lambda_{CC}\right] = \omega_k a_k^\dagger a_k + \mathcal{H}_{CC} - \Lambda_{CC}.$$

So $\mathcal{E}_{\text{total}} = \mathcal{H}_{CC} - \Lambda_{CC}$.
(d) $\mathcal{H}_{CC} \sim M_{pl}^4 \sim (10^{19}\ \text{GeV})^4$. So the amount of fine-tuning required is

$$\frac{\mathcal{H}_{CC} - \Lambda_{CC}}{\mathcal{H}_{CC} + \Lambda_{CC}} \sim \left(\frac{10^{-3}\ \text{eV}}{10^{19}\ \text{GeV}}\right)^4 \sim 10^{-124}.$$

In other words, the two number must cancel to one in 124 digits in order to result in the observed vaccumm energy density!