Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html

SOLUTIONS FOR ASSIGNMENT #4

Reading Assignments:
Section 3.2 of Peskin and Schroeder.

Problem 1
Do Problem 3.4 in Peskin and Schroeder, but leave out part (e).
Solution:
(a) According to Eq. (37) in Peskin and Schroeder, and using the identity $\vec{\sigma}^* = -\sigma^2\vec{\sigma}\sigma^2$, the χ spinor has the following property under Lorentz transformation Λ:

$$\chi(x) \to e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}-i\vec{\beta})}\chi(\Lambda^{-1}x),$$

$$i\sigma^2\chi^*(x) \to e^{i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}+i\vec{\beta})}\ \left(i\sigma^2\chi^*(\Lambda^{-1}x)\right).$$

Then from Problem 3.1(c) in Peskin and Schroeder we have proven the following identity for $\bar{\sigma}^\mu$:

$$e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}+i\vec{\beta})}\bar{\sigma}^\mu e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}-i\vec{\beta})} = \Lambda^\mu{}_\nu\bar{\sigma}^\nu,$$

which implies

$$\begin{aligned} i\bar{\sigma}\cdot\partial\chi(x) &\to i\bar{\sigma}\cdot\partial\ e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}-i\vec{\beta})}\chi(\Lambda^{-1}x) \\ &= e^{i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}+i\vec{\beta})}i\bar{\sigma}^\mu\ (\Lambda^\mu{}_\nu\partial^\nu)\chi(\Lambda^{-1}x). \end{aligned}$$

Therefore we have shown that the equation

$$\left[i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi(x)\right] \to e^{i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta}+i\vec{\beta})}\left[i\bar{\sigma}^\mu\ (\Lambda^\mu{}_\nu\partial^\nu)\chi(\Lambda^{-1}x) - im\sigma^2\chi(\Lambda^{-1}x)\right]$$

is relativistically invariant. To show that the equation also implies Klein-Gordon equation, simply take the complex-conjugate equation

$$\chi = \frac{1}{m}(\sigma^2)^*(\bar{\sigma})^*\cdot\partial\chi^*$$

and plug back into the original equation to arrive at $(\partial^2+m^2)\chi^*=0$.
(b) The action is

$$S = \int d^4x\left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right].$$

Under complex-conjugation, the kinetic term becomes

$$\int d^4x(\chi^\dagger i\bar{\sigma}\cdot\partial\chi)^* = \int d^4x(-i)(\partial_\mu\chi^\dagger)\bar{\sigma}^\mu\chi = \int d^4x\chi^\dagger i\bar{\sigma}\cdot\partial\chi$$

where we have used the Grassmann nature of the Majorana spinor

$$(\alpha\beta)^* = -\alpha^*\beta^*$$

as well as integration by part in the last step. The reality of the mass term in the action also follows from the Grassmann variable. It is straightforward to see that $\delta S/\delta\chi^\dagger$ gives the Majorana equation.
(c) For $\psi_D = (\psi_L, \psi_R)^T$,

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi = i\psi_L^\dagger\bar{\sigma}\cdot\partial\psi_L + i\psi_R^\dagger\sigma\cdot\partial\psi_R - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$$

Now if we write $\psi_L = \chi_1$ and $\psi_R = i\sigma^2\chi_2^*$,

$$\mathcal{L}_D = i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_2^T\sigma^2(\sigma\cdot\partial)\sigma^2\chi_2^* - im(\chi_1^\dagger\sigma^2\chi_2^* - \chi_2^T\sigma^2\chi_1),$$

where

$$\chi_2^T\sigma^2(\sigma\cdot\partial)\sigma^2\chi_2^* = \left(\chi_2^T\sigma^2(\sigma\cdot\partial)\sigma^2\chi_2^*\right)^T = \chi_2^\dagger\bar{\sigma}\cdot\partial\chi_2.$$

So the Lagrangian is

$$\mathcal{L}_D = i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_2^\dagger\bar{\sigma}\cdot\partial\chi_2 - im(\chi_1^\dagger\sigma^2\chi_2^* - \chi_2^T\sigma^2\chi_1).$$

(d) The action has in (c) has a global symmetry $\chi_1 \to e^{i\alpha}\chi_1$ and $\chi_2 \to e^{-i\alpha}\chi_2$. The current corresponding the this global symmetry is $J^\mu = \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2$. This current is conserved $\partial_\mu J^\mu = 0$. The action in (b) has a global symmetry $\chi \to e^{i\alpha}\chi$ if the mass term is absent $m = 0$. The current is $J^\mu = \chi^\dagger\bar{\sigma}\chi$, whose divergence is proportional to the mass term $\partial_\mu J^\mu = m(\chi^T\sigma^2\chi + \chi^\dagger\sigma^2\chi^*)$. A theory of N massive 2-component fermions with $O(N)$ symmetry can be written as

$$S = \sum_{i=1}^{N}\int d^4x\ i\chi_i^\dagger\bar{\sigma}\cdot\partial\chi_i + \frac{im}{2}\left(\chi_i^T\sigma^2\chi_i - \chi_i^\dagger\sigma^2\chi_i^*\right).$$

Problem 2
Do Problem 3.5 in Peskin and Schroeder.
Solution:
(a) The Lagrangian

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F$$

is known as the Wess-Zumino model and the simplest $\mathcal{N} = 1$ SUSY theory in four dimensions. The variation under a infinitesimal global SUSY transformation δ_ϵ is

$$\begin{aligned}
&\delta_\epsilon\phi = -i\epsilon^T\sigma^2\chi, && \delta_\epsilon\phi^* = +i\chi^\dagger\sigma^2\epsilon^*,\\
&\delta_\epsilon\chi = \epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^*, && \delta_\epsilon\chi^\dagger = \epsilon^\dagger F^* + \epsilon^T\sigma^2\sigma\cdot\partial\phi^*,\\
&\delta_\epsilon F = -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi, && \delta_\epsilon F^* = +i(\partial_\mu\chi^\dagger)\bar{\sigma}^\mu\epsilon.
\end{aligned}$$

(A global transformation means the two-component Grassmann spinor is constant.) From the above one can compute

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta_\epsilon\phi}\delta_\epsilon\phi + (\text{c.c.}) &= +i\epsilon^T\sigma^2\chi\partial^2\phi^* - i\chi^\dagger\sigma^2\epsilon^*\partial^2\phi && (1)\\
\frac{\delta\mathcal{L}}{\delta_\epsilon\chi}\delta_\epsilon\chi + (\text{c.c.}) &= \epsilon^\dagger(i\bar\sigma\cdot\partial\chi)F^* - i(\partial_\mu\chi^\dagger)\bar\sigma^\mu\epsilon F\\
&\quad +\epsilon^T\sigma^2\sigma\cdot\partial\phi^*(i\bar\sigma\cdot\partial\chi) - i(\partial_\mu\chi^\dagger)\bar\sigma^\mu\sigma\cdot\partial\phi\sigma^2\epsilon^* && (2)\\
\frac{\delta\mathcal{L}}{\delta_\epsilon F}\delta_\epsilon F + (\text{c.c.}) &= -i\epsilon^\dagger\bar\sigma\cdot\partial\chi F^* + i(\partial_\mu\chi^\dagger)\bar\sigma^\mu\epsilon F. && (3)
\end{aligned}$$

It is easy to see that the first line in Eq. (2) cancels Eq. (3). On the other hand, the second line in Eq. (2) can be re-written, using the Leibniz rule of derivative and up to total derivatives, as

$$-i\epsilon^T\sigma^2\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu\phi^*\chi + i\chi^\dagger\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu\phi\sigma^2\epsilon^*$$

We then need to compute $\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu = (1/2)\{\sigma^\mu,\bar\sigma^\nu\}\partial_\mu\partial_\nu$. Using the definitions $\sigma^\mu = (1,\vec\sigma)$ and $\bar\sigma^\mu = (1,-\vec\sigma)$, as well as $\{\sigma^a,\sigma^b\} = 2\delta^{ab}$ for Pauli matrices, one can verify that

$$\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\sigma^\mu,\bar\sigma^\nu\}\partial_\mu\partial_\nu = \partial^2$$

Then one sees immediately the second line in Eq. (2) cancels Eq. (1), up to total derivatives. The Lagrangian is therefore invariant.

(b) The interacting part of the Wess-Zumino model is

$$\Delta\mathcal{L} = \left(m\phi F + \frac{1}{2}im\chi^T\sigma^2\chi\right) + (\text{c.c.})$$

One computes the variation of $\Delta\mathcal{L}$ as

$$-im\epsilon^T\sigma^2\chi F - im\phi\epsilon^\dagger\bar\sigma\cdot\partial\chi + \frac{i}{2}m\left[(\epsilon^T F - \epsilon^\dagger\bar\sigma\cdot\partial\phi\sigma^2)\sigma^2\chi + \chi^T\sigma^2(\epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^*)\right] + (\text{c.c.})$$

It is straightforward to verify that the last term gives exactly the same contribution as the third term, and that the whole variation is zero up to a total derivative.

The equation of motion for the auxiliary field F is $F = -m\phi^*$ and $F^* = -m\phi$. Plug it back into the Lagrangian we see the fermion and the boson have the same mass.

(c) We only have to show the part involving the superpotential is invariant under SUSY transformation:

$$\begin{aligned}
\delta_\epsilon\mathcal{L}_W &= -i\epsilon^\dagger\bar\sigma\cdot\partial\chi_i\frac{\partial W}{\partial\phi_i} + \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}F_i(-i\epsilon^T\sigma^2\chi_j)\\
&\quad +\frac{i}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\left[(\epsilon^T F_i - \epsilon^\dagger\bar\sigma\cdot\partial\phi_i\sigma^2)\sigma^2\chi_j + \chi_i^T\sigma^2(\epsilon F_j + \sigma\cdot\partial\phi_j\sigma^2\epsilon^*)\right]\\
&\quad +\frac{i}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_i^T\sigma^2\chi_j(-i\epsilon^T\sigma^2\chi_k), && (4)
\end{aligned}$$

where the first term is equivalent to, up to a total derivative,

$$+i\epsilon^\dagger\bar\sigma\cdot\partial\phi_j\chi_i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}$$

so in the end the first two lines in Eq. (4) cancel up to a total derivative, in a similar fashion to (b). The last line, however, is zero identically because both χ_i and ϵ are two-component Grassmann spinors. More explicitly, the last term is

$$\frac{1}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}(\chi_1\chi_2 - \chi_2\chi_1)(\epsilon_1\chi_2 - \epsilon_2\chi_1),$$

which contains either $\chi_i\chi_i = -\chi_i\chi_i = 0$.

The equation of motion for the auxiliary field with the superpotential is $F^* = -\partial W/\partial\phi_i$ so the Lagrangian now becomes

$$\mathcal{L} = \partial_\mu\phi_i^*\partial^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}\cdot\partial\chi_i - \left|\frac{\partial W[\phi]}{\partial\phi_i}\right|^2 + \frac{i}{2}\frac{\partial W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j + \text{c.c.} \quad .$$

The equations of motion for ϕ_i and χ_i are

$$\begin{aligned} \partial^2\phi_i &= \frac{\partial}{\partial\phi_i^*}\left|\frac{\partial W[\phi]}{\partial\phi_i}\right|^2 - \frac{i}{2}\frac{\partial W[\phi]}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_j^T\sigma^2\chi_k + \text{c.c.} \\ i\bar{\sigma}\cdot\partial\chi_i &= \frac{i}{2}\left(\frac{\partial W[\phi]}{\partial\phi_i\partial\phi_j}\right)^*\sigma^2\chi_j^*. \end{aligned}$$

With the above expressions it is simple to get the equations for motion for $n = 1$ and $W = g\phi^3/3$.

Problem 3

(a) In the class we showed that the conserved currents corresponding to spacetime translations $x^\alpha \to x^\alpha - a^\alpha$ are the energy-momentum tensor $T^{\mu\nu}$. Since we have been considering Lorentz-invariant quantum field theories, derive the conserved currents corresponding to infinitesimal Lorentz transformations $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta$.
(Hint: recall that in the case of translations, there are really four currents $T^{\mu\alpha} \equiv (j^\mu)^\alpha$, one for each a^α. In this case there are really six conserved currents $M_\beta^{\mu\alpha} \equiv (j^\mu)^\alpha{}_\beta$, one for each $\omega^\alpha{}_\beta$. You may wish to express $M_\beta^{\mu\alpha}$ in terms of $T^{\mu\alpha}$.)
(b) What is the physical interpretation for each of the conserved charges in (a)? Separate your discussions into those corresponding to rotations and those corresponding to Lorentz boosts.
Solution: (a) Under the Lorentz transformation a scalar field

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) = \phi(x^\mu - \omega^\mu{}_\nu x^\nu) = \phi(x^\mu) - \omega^\mu{}_\nu x^\nu\partial_\mu\phi(x).$$

Since the Lagrangian is also a scalar, we obtain that $\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu\partial_\mu\mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu\mathcal{L})$, which is a total derivative. The Noether's theorem gives the conserved current in this case

$$\begin{aligned} j^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\omega_\nu^\rho x^\nu\partial_\rho\phi + \omega^\mu{}_\nu x^\nu\mathcal{L} \\ &= -\omega_\nu^\rho\left[-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}x^\nu\partial_\rho\phi - \delta^\mu\rho x^\nu\mathcal{L}\right] = -\omega_\nu^\rho T_\rho^\mu x^\nu, \end{aligned} \tag{5}$$

where $T^{\mu\nu}$ is the stress-energy tensor. Since there are six independent infinitesimal transformations ($\omega_{\mu\nu}$ is anti-symmetric!), there are really six currents, one for each independent $\omega_{\mu\nu}$:

$$(\mathcal{T}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

which satisfy $\partial_\mu(\mathcal{T}^\mu)^{\rho\sigma}=0$.
(b) For spatial rotations $\{\rho,\sigma\}=\{1,2,3\}$ and the three conserved charges are

$$Q^{ij}=\int d^3x(x^iT^{0j}-x^jT^{0i})$$

which give the total angular momentum of the field. For the three boost the three charges are

$$Q^{ii}=\int d^3x(x^0T^{0i}-x^iT^{00}).$$

The fact that they are conserved implies

$$\frac{dQ^{0i}}{dt}=\int d^3xT^{0i}+t\int d^3x\frac{dT^{0i}}{dt}-\frac{d}{dt}\int d^3xx^iT^{00}=P^i+t\frac{dP^i}{dt}-\frac{d}{dt}\int d^3xx^iT^{00}.$$

Since the momentum P^i is already conserved $dP^i/dt=0$, we have the following time-invariant quantity under boost

$$\frac{d}{dt}\int d^3xx^iT^{00}=\text{constant}.$$

This is a statement that the centre-of-energy of the field travels at constant velocity. In Newtonian dynamics the corresponding statement is the centre-of-mass of a system travels at constant velocity.

Problem 4
(a) A Lorentz transformation $\Lambda^\mu{}_\nu$ leaves the metric tensor $g_{\mu\nu}$ invariant: $\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\, g_{\mu\nu}=g_{\alpha\beta}$. Use this equation to prove that $\Lambda^0{}_0\geq 1$ or $\Lambda^0{}_0\leq -1$.
(b) Show that if two Lorentz transformations Λ_1 and Λ_2 both have $(\Lambda_1)^0{}_0\geq 1$ and $(\Lambda_2)^0{}_0\geq 1$, then $\Lambda_3=\Lambda_1\Lambda_2$ also has $(\Lambda_3)^0{}_0\geq 1$. In other words, this sign is preserved under Lorentz group action and can be used to classify Lorentz transformations.
(c) Show that if two Lorentz transformations Λ_1 and Λ_2 both have $\mathrm{Det}(\Lambda_1)>0$ and $\mathrm{Det}(\Lambda_2)>0$, then $\Lambda_3=\Lambda_1\Lambda_2$ also has $\mathrm{Det}(\Lambda_3)>0$. In other words, this sign is preserved under Lorentz group action and can be used to classify Lorentz transformations.
(d) Show all Lorentz transformations with $\mathrm{Det}(\Lambda)>0$ and $\Lambda^0{}_0\geq 1$ form a subgroup of the Lorentz group.
Solution:
(a) The 00 component of $\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma g_{\mu\nu}=g_{\rho\sigma}$ implies

$$1=g_{00}=\Lambda^\mu{}_0\Lambda^\nu{}_0=(\Lambda^0{}_0)^2-\Lambda^i{}_0\Lambda^i{}_0,$$

from which we see either $\Lambda^0{}_0\geq 1$ or $\Lambda^0{}_0\leq -1$.
(b) Define two three-vectors $\mathbf{X}_i=\Lambda_1{}^0{}_i$ and $\mathbf{Y}^i=\Lambda_2{}^i{}_0$. From (a) we see that $|\mathbf{X}|=\sqrt{(\Lambda_1{}^0{}_0)^2-1}$ and $|\mathbf{Y}|=\sqrt{(\Lambda_2{}^0{}_0)^2-1}$. On the other hand, $(\Lambda_3)^0{}_0=\Lambda_1{}^0{}_0\Lambda_2{}^0{}_0-\mathbf{X}\cdot\mathbf{Y}$. Since $\mathbf{X}\cdot\mathbf{Y}\leq|\mathbf{X}||\mathbf{Y}|$, we arrive at

$$(\Lambda_3)^0{}_0\geq\Lambda_1{}^0{}_0\Lambda_2{}^0{}_0-\sqrt{(\Lambda_1{}^0{}_0)^2-1}\sqrt{(\Lambda_2{}^0{}_0)^2-1}\geq 1.$$

(c) The statement follows from $\mathrm{Det}(\Lambda_3)=\mathrm{Det}(\Lambda_1)\mathrm{Det}(\Lambda_2)$.
(d) The statement follows from (a), (b), and (c).