Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

ASSIGNMENT #5

Due at 3:30 PM, October 31st

(Two pages and three problems.)

Reading Assignments:
Sections 3.3 to 3.5 of Peskin and Schroeder.

Problem 1
Do Problem 3.6 in Peskin and Schroeder.

Problem 2
In non-relativistic quantum mechanics, the Schrodinger equation implies a continuity equation of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

where $\rho = |\psi|^2$ is interpreted as the probability dentiy.
(a) Derive a similar continuity equation for the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi = 0.$$

Write down ρ and $\vec{j}$ explicitly. Explain what would go wrong if you were to interpret ρ as the probability density.
(b) Dirac realized in 1928 while staring at a fireplace in St. John's College, Cambridge, that the sickness in (a) is the result of an equation of motion that is second order in time derivative. He thus proposed instead

$$i\frac{\partial \psi}{\partial t} = H_D \psi \tag{1}$$

Explain why the Hamiltonian H_D here must be first order in spatial derivatives and contain only terms linear in the mass m.
(c) Dirac then guessed a general form of H_D

$$H_D = -i a_i \frac{\partial}{\partial x^i} + a_4 m.$$

H_D must be such that when applying Eq. (1) twice one recovers the Klein-Gordon equation. (After all, $E^2 - |\vec{p}|^2 = m^2$!) Firsts show that the set $\{a_i, i = 1, 2, 3, 4\}$ cannot be pure numbers, then derive the conditions a_i must satisfy to produce Klein-Gordon equation. Rewrite $\gamma^0 = a_4$ and $\gamma^i = a_4 a_i$ and re-express your coditions in terms of γ^μ.
(d) What is the continuity equation following from the Dirac equation? Give ρ and $\vec{j}$

explicitly. Can you interpret ρ as the probability density?
(e) As we showed in class the Dirac equation has solutions of the plane-wave form

$$\psi \propto e^{\pm ik\cdot x}.$$

Explain why this is a problem when trying to consider the Dirac equation as quantum mechanical. Then explain why this is *not* a problem when promoting the Dirac equation to a quantum field theory.

Problem 3
In class we defined a Dirac spinor ψ in the chiral basis:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where ψ_L and ψ_R are Weyl spinors transforming under the Lorentz group according to Eq. (3.37) in Peskin and Schroeder. The Dirac Lagrangian is then written as

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + i\psi^\dagger \vec{\alpha} \cdot \nabla \psi - m\psi^\dagger \beta \psi$$

where $\vec{\alpha}$ and β are 4×4 matices.
(a) Write out $\vec{\alpha}$ and β explicitly and show they satisfy the same conditions for $\{a_i, i = 1, 2, 3, 4\}$ you work out in Problem 2(c) if you identify $\alpha^i = a^i, i = 1, 2, 3$ and $\beta = a_4$. Then go to a different basis, the Dirac basis, for a Dirac spinor

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix}$$

and write out $\vec{\alpha}$ and β in this basis. Show that in this basis $\vec{\alpha}$ and β still satisfy the same conditions as in the chiral basis.
(b) Stay in the chiral basis, work out how a Dirac spinor transform under rotation and boost, respectively. Write out the generators for the rotation and the boost explicitly, from which deduce the generators $S^{\mu\nu}$ for the Lorentz group. Moreover, show that

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

where the Dirac gamma matrices are defined in Problem 2(c).
(c) Show that γ^μ transform like a vector under the infinitesimal Lorentz transformation $D(\Lambda) = 1 - i\omega_{\mu\nu} S^{\mu\nu}/2$. In other words, show

$$D(\Lambda)^{-1} \gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$$

(Hint: you can prove the statement by computing a certain commutator.)
(d) Explain why all the results in (b) and (c) would still hold in a different basis such as the Dirac basis.