Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

Ian Low, Fall 2008

Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTONS FOR ASSIGNMENT #5

Reading Assignments:
Sections 3.3 to 3.5 of Peskin and Schroeder.

Problem 1
Do Problem 3.6 in Peskin and Schroeder.
Solution:
(a) Let $\Gamma^A = \{1, \tilde{\gamma}^\mu, \tilde{\sigma}^{\mu\nu}(\mu < \nu), i\gamma^5\tilde{\gamma}^\mu, \gamma^5\}$, where

$$\tilde{\gamma}^\mu = \{\gamma^0, i\gamma^j\}, \quad \tilde{\sigma}^{\mu\nu} = \{i\sigma^{0j}, \sigma^{ij}\},$$

then it is easy to see that there are sixteen of them and $\mathrm{Tr}(\Gamma^A\Gamma^B) = 4\delta^{AB}$.
(b) Since any 4×4 matrix M can be written as a linear combination of Γ^A: $M = \sum_A c_A \Gamma^A$, the orthogonality condition in (a) implies $c_A = \mathrm{Tr}(M\Gamma^A)/4$ and $M = \sum_A \mathrm{Tr}(M\Gamma^A)\Gamma^A/4$, which, in component form becomes

$$M_{ij} = \frac{1}{4}\sum_A M_{kl}(\Gamma^A)_{lk}(\Gamma^A)_{ij}.$$

Therefore we arrive at the following closure relation

$$\frac{1}{4}\sum_A (\Gamma^A)_{lk}(\Gamma^A)_{ij} = \delta_{lj}\delta_{ki}.$$

Multiply the closure relation by $(\Gamma^B)_{ml}(\Gamma^C)_{ni}$ to get

$$\begin{aligned}(\Gamma^B)_{mj}(\Gamma^C)_{nk} &= \frac{1}{4}\sum_A (\Gamma^B\Gamma^A)_{mk}(\Gamma^C\Gamma^A)_{nj} \\ &= \frac{1}{4}\sum_A \frac{1}{4}\mathrm{Tr}(\Gamma^B\Gamma^A\Gamma^D)(\Gamma^D)_{mk}\,\frac{1}{4}\mathrm{Tr}(\Gamma^C\Gamma^A\Gamma^E)(\Gamma^E)_{nj}, \qquad (1)\end{aligned}$$

where we have used the orthogonality condition to expand the product $\Gamma^M\Gamma^N = \mathrm{Tr}(\Gamma^M\Gamma^N\Gamma^P)\Gamma^P$. Then use the closure relation again to compute the product of the trace

$$\sum_A \frac{1}{4}\mathrm{Tr}(\Gamma^B\Gamma^A\Gamma^D)\,\frac{1}{4}\mathrm{Tr}(\Gamma^C\Gamma^A\Gamma^E) = \frac{1}{4}\mathrm{Tr}(\Gamma^B\Gamma^E\Gamma^C\Gamma^D).$$

In the end we obtain

$$(\Gamma^B)_{mj}(\Gamma^C)_{nk} = \frac{1}{16}\mathrm{Tr}(\Gamma^D\Gamma^B\Gamma^E\Gamma^C)(\Gamma^D)_{mk}(\Gamma^E)_{nj}.$$

(c) For $\Gamma^B = \Gamma^C = \mathbb{1}$, $\mathrm{Tr}(\Gamma^D \mathbb{1} \Gamma^E \mathbb{1}) = 4\delta^{DE}$. Thus

$$(\bar{u}_1 u_2)(\bar{u}_3 u_4) = \frac{1}{4}\sum_A (\bar{u}_1 \Gamma^A u_4)(\bar{u}_3 \Gamma^A u_2).$$

For $\Gamma^B = \gamma^\mu$ and $\Gamma^C = \gamma_\mu$, it is straightforward to verify the following identities

$$\gamma^\mu \gamma_\mu = 4, \quad \gamma^\mu \tilde{\gamma}_\nu \gamma_\mu = -2\tilde{\gamma}_\nu, \quad \gamma^\mu \tilde{\sigma}^{\rho\tau} \gamma_\mu = 0, \quad \gamma^\mu (i\gamma^5 \tilde{\gamma}^\nu)\gamma_\mu = 2(i\gamma^5\tilde{\gamma}^\nu), \quad \gamma^\mu \gamma^5 \gamma_\mu = -4\gamma^5,$$

from which we get

$$\begin{aligned}(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4) \;=\; & \frac{1}{4}\left[4(\bar{u}_1 u_4)(\bar{u}_3 u_2) - 2(\bar{u}_1 \gamma^\mu u_4)(\bar{u}_3 \gamma_\mu u_2)\right. \\ & \left. -2(\bar{u}_1 \gamma^5 \gamma^\mu u_4)(\bar{u}_3 \gamma^5 \gamma_\mu u_2) - 4(\bar{u}_1 \gamma^5 u_4)(\bar{u}_3 \gamma^5 u_2)\right], \end{aligned} \tag{2}$$

where we have used the fact that $(\bar{u}_1 \tilde{\gamma}^\mu u_4)(\bar{u}_3 \tilde{\gamma}^\mu u_2) = (\bar{u}_1 \gamma^\mu u_4)(\bar{u}_3 \gamma_\mu u_2)$.

Problem 2

In non-relativistic quantum mechanics, the Schrodinger equation implies a continuity equation of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

where $\rho = |\psi|^2$ is interpreted as the probability dentiy.

(a) Derive a similar continuity equation for the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi = 0.$$

Write down ρ and $\vec{j}$ explicitly. Explain what would go wrong if you were to interpret ρ as the probability density.

(b) Dirac realized in 1928 while staring at a fireplace in St. John's College, Cambridge, that the sickness in (a) is the result of an equation of motion that is second order in time derivative. He thus proposed instead

$$i\frac{\partial \psi}{\partial t} = H_D \psi \tag{3}$$

Explain why the Hamiltonian H_D here must be first order in spatial derivatives and contain only terms linear in the mass m.

(c) Dirac then guessed a general form of H_D

$$H_D = -ia_i \frac{\partial}{\partial x^i} + a_4 m.$$

H_D must be such that when applying Eq. (3) twice one recovers the Klein-Gordon equation. (After all, $E^2 - |\vec{p}|^2 = m^2$!) Firsts show that the set $\{a_i, i = 1, 2, 3, 4\}$ cannot be pure numbers, then derive the conditions a_i must satisfy to produce Klein-Gordon equation. Re-write $\gamma^0 = a_4$ and $\gamma^i = a_4 a_i$ and re-express your coditions in terms of γ^μ.

(d) What is the continuity equation following from the Dirac equation? Give ρ and $\vec{j}$ explicitly. Can you interpret ρ as the probability density?

(e) As we showed in class the Dirac equation has solutions of the plane-wave form

$$\psi \propto e^{\pm ik\cdot x}.$$

Explain why this is a problem when trying to consider the Dirac equation as quantum mechanical. Then explain why this is *not* a problem when promoting the Dirac equation to a quantum field theory.

Solution:

(a) For the Klein-Gordon equation $\rho = \psi^* \partial_t \psi - (\partial_t \psi^*)\psi$ and $j^i = \psi^* \partial_i \psi - (\partial_i \psi^*)\psi$. They satisfy $\partial_t \rho + \nabla \cdot \vec{j} = 0$. Since ρ is not positive-definte, one cannot interpret ρ as the probability density.

(b) Since the equation of motion is first-order in time derivative, by special relativity H_D must also be first-order in spatial derivatives and mass.

(c)

$$-\frac{\partial^2 \phi}{\partial t^2} = H_D^2 = -a_i a_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} - im\{a_i, a_4\}\frac{\partial \phi}{\partial x_i} + m^2 a_4^2 \psi$$

In order to reproduce the Klein-Gordon equation, the cross term $\partial_i \partial_j \psi$ as well as $\partial_i \psi$ must be zero, which implies $\{a_i, a_j\} = 0$ for $i, j = 1, 2, 3, 4$. Therefore the a_is cannot be pure numbers. The conditions for the above equation to turn into Klein-Gordon equation is

$$\{a_i, a_j\} = 2\delta_{ij}, \quad \{a_i, a_4\} = 0, \quad a_4^2 = 1.$$

In terms of $\gamma^0 = a_4, \gamma^i = a_4 a_i$, the above condition becomes $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$.

(d) For the Dirac equation $\rho = \psi^\dagger \psi$ and $j^i = \psi^\dagger a_i \psi$ which satisfies $\partial_t \rho + \nabla \cdot \vec{j} = 0$. Since ρ is positive-definite, it is possible to interpret ρ as the probability density.

(e) This is a problem quantum mechanically because one of the plane wave solutions can only be interpreted as a negative energy solution and the energy is unbounded from below. As a field theory the plane wave solutions are interpreted as creating and annihilating a particle, and as such is not a pathology.

Problem 3

In class we defined a Dirac spinor ψ in the chiral basis:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where ψ_L and ψ_R are Weyl spinors transforming under the Lorentz group according to Eq. (3.37) in Peskin and Schroeder. The Dirac Lagrangian is then written as

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + i\psi^\dagger \vec{\alpha} \cdot \nabla \psi - m\psi^\dagger \beta \psi$$

where $\vec{\alpha}$ and β are 4×4 matices.

(a) Write out $\vec{\alpha}$ and β explicitly and show they satisfy the same conditions for $\{a_i, i = 1, 2, 3, 4\}$ you work out in Problem 2(c) if you identify $\alpha^i = a^i, i = 1, 2, 3$ and $\beta = a_4$. Then go to a different basis, the Dirac basis, for a Dirac spinor

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix}$$

and write out $\vec{\alpha}$ and β in this basis. Show that in this basis $\vec{\alpha}$ and β still satisfy the same conditions as in the chiral basis.

(b) Stay in the chiral basis, work out how a Dirac spinor transform under rotation and boost, respectively. Write out the generators for the rotation and the boost explicitly, from which deduce the generators $S^{\mu\nu}$ for the Lorentz group. Moreover, show that

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

where the Dirac gamma matrices are defined in Problem 2(c).
(c) Show that γ^μ transform like a vector under the infinitesimal Lorentz transformation $D(\Lambda) = 1 - i\omega_{\mu\nu}S^{\mu\nu}/2$. In other words, show

$$D(\Lambda)^{-1}\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$$

(Hint: you can prove the statement by computing a certain commutator.)
(d) Explain why all the results in (b) and (c) would still hold in a different basis such as the Dirac basis.
Solution:
(a) In the Weyl basis

$$\alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \beta = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix}.$$

They satisfy the commutation relations $\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \{\alpha^i, \beta\} = 0$, and $\beta^2 = 1$. In the Dirac basis,

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}.$$

Since this is an orthogonal rotation of basis, the commutation relation is unchanged.
(b) From the transformation laws of Weyl spinors in Eqs. (3.37) in Peskin and Schroeder, one can see that

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \begin{pmatrix} e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}} & 0 \\ 0 & e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}} \end{pmatrix} \psi_D.$$

The generators of rotation $\vec{L}$ and boost $\vec{M}$ are

$$\vec{L} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \vec{M} = -\frac{i}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

In terms of the generators of the Lorentz group $S^{i0} = M^i$ and $S^{ij} = \epsilon^{ijk}L^k$. With the Dirac matrices defined in Problem 2(c), it is straightforward to show that $S^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$.
(c) All we need to show is

$$[\gamma^\mu, S^{\rho\tau}] = (\mathcal{T}^{\rho\tau})^\mu_\nu \gamma^\nu$$

where $(\mathcal{T}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$.
(d) A change of basis is just an orthogonal rotation, which will not change all the results in (b) and (c).