Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTIONS FOR ASSIGNMENT #6

Reading Assignments:
Sections 3.6 of Peskin and Schroeder.

Problem 1
Do Problem 3.2 in Peskin and Schroeder.
Solution:
The spinors satisfy the Dirac equation $u(p) = \not{p}u(p)/m$ and $\bar{u}(p') = \bar{u}(p')\not{p}'/m$, from which we have

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}[\bar{u}(p')\gamma^\mu \not{p}u(p) + \bar{u}(p')\not{p}'\gamma^\mu u(p)].$$

Then use the identity $\gamma^\mu\gamma^\nu = (1/2)\{\gamma^\mu, \gamma^\nu\} + (1/2)[\gamma^\mu, \gamma^\nu]$ one can show the above equation leads to

$$\bar{u}(p') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'_\mu - p_\mu)}{2m} \right] u(p).$$

Problem 2
We are going to consider the Lorentz transformation property of the Dirac bilinear $\Psi^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$. Because $\Psi^{\mu\nu}$ is anti-symmetric, there are six independent components in it. On the other hand, an irreducible representation (j_1, j_2) of $SO(1,3)$ has $(2j_1 + 1) \times (2j_2 + 1)$ independent components. (This is called the dimension of the representation.)
(a) Write down all the irreducible representations of $SO(1,3)$ that are six-dimensional. Show that $\Psi^{\mu\nu}$ can be none of them. (Hint: What is the parity of $\Psi^{\mu\nu}$?)
(b) Next we need to consider reducible representations of the form $(j_1, j_2) \oplus (j'_1, j'_2)$. By counting the dimensionality show that there is a unique reducible representation in which $\Psi^{\mu\nu}$ could possibly fit.
(c) Recall in classical electromagnetism we define the dual of the field strength as $\tilde{F}_{\mu\nu} = (1/2)\epsilon^{\mu\nu\delta\rho}F_{\delta\rho}$. Let's define

$$\Psi^{\mu\nu}_{\pm} = \frac{1}{2}(\Psi^{\mu\nu} \pm i\tilde{\Psi}^{\mu\nu}).$$

Then we can decompose $\Psi^{\mu\nu} = \Psi^{\mu\nu}_+ + \Psi^{\mu\nu}_-$. Show that $\Psi^{\mu\nu}_\pm$ are *self-dual* tensors which satisfy

$$\tilde{\Psi}^{\mu\nu}_\pm = \mp i\Psi^{\mu\nu}_\pm.$$

Again use the dimensionality counting to show that $\Psi^{\mu\nu}_\pm$ would each fits into a unique irreducible representation of the Lorentz group.
(d) Now prove your answers in (b) and (c) by explicitly working out the transformation property of $\Psi^{\mu\nu}$ under the Lorentz group. (Hint: as always, to see how an object O transforms under the Lorentz group, you compute the commutator of O with the generators of

$SO(1,3)$.)
Solution:
(a) The dimensionality of (j_1, j_2) is $(2j_1+1)\times(2j_2+1)$, so the irreducible six-dimensional representations are $(0,5/2), (5/2,0), (1/2,1)$, and $(1,1/2)$. None of them are parity-invariant, so $\Psi^{\mu\nu}$ cannot sit in any of them.
(b) Parity invariance requires the reducible representation to be of the form $(j_1,j_2)\oplus(j_2,j_1)$, whose dimensionality is $2\times(2j_1+1)\times(2j_2+1)$. Thus there is only one reducible representation that is both six-dimensional and parity-invariant, which is $(1,0)\oplus(0,1)$.
(c) A general rank-two anti-symmetric tensor has six independent components:

$$F^{\mu\nu} = \begin{pmatrix} 0 & F^{01} & F^{02} & F^{03} \\ & 0 & F^{12} & F^{13} \\ & & 0 & F^{23} \\ & & & 0 \end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & F^{23} & -F^{13} & F^{12} \\ & 0 & -F^{03} & F^{02} \\ & & 0 & -F^{01} \\ & & & 0 \end{pmatrix}.$$

Self-duality reduces the number down to three:

$$F^{\mu\nu}_{\pm} = \frac{1}{2}(F^{\mu\nu} \pm i\tilde{F}^{\mu\nu}) = \frac{1}{2}\begin{pmatrix} 0 & F^{01} \pm iF^{23} & F^{02} \mp iF^{13} & F^{03} \pm iF^{12} \\ & 0 & \mp i(F^{03} \pm iF^{12}) & \pm i(F^{02} \mp iF^{13}) \\ & & 0 & \mp i(F^{01} \pm iF^{23}) \\ & & & 0 \end{pmatrix}.$$

So $\Psi^{\mu\nu}_{\pm}$ must be fit into either (1,0) or (0,1).
(d) The crucial observation here is that the generator of the Lorentz group $S^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ is related to $\sigma^{\mu\nu} = 2S^{\mu\nu}$. So if we define

$$\begin{aligned} L^i &= \frac{1}{2}\epsilon^{ijk}S^{jk}, \quad K^i = S^{0i}, \quad J^i_{\pm} = \frac{1}{2}(L \pm iK^i), \\ L^i_{\Psi} &= \frac{1}{2}\epsilon^{ijk}\sigma^{jk}, \quad K^i_{\Psi} = \sigma^{0i}, \quad (J_\Psi)^i_{\pm} = \frac{1}{2}(L^i_\Psi \pm iK^i_\Psi) \end{aligned}$$

then from Problem 3.1 of Peskin and Schroeder we know immediately that

$$[J^i_{\pm}, (J_\Psi)^j_{\pm}] = i\epsilon^{ijk}(J_\Psi)^k_{\pm}.$$

In other words, $(J_\Psi)_+$ transform like $(1,0)$ (i.e. a three-vector under $SU(2)_+$) and $(J_\Psi)_-$ like (0,1). Therefore $\Psi^{\mu\nu}$ transform like $(1,0)\oplus(0.1)$. Furthermore, from the explicit expressions for $F^{\mu\nu}_{\pm}$ in (c) it is easy to see that $(J_\Psi)^i_{\pm} = \pm i\Psi^{0i}_{\mp}$.

Problem 3
(a) Show that the Dirac Lagrangian is invariant under space-time translations: $x^\mu \to x^\mu + a^\mu$.
(b) Derive the energy-momentum tensor of the Dirac field – the Noether current corresponding to translations.
(c) Construct the expression for the conserved physical momentum $\mathbf{P}$ in the quantum theory (that is, express it in terms of raising and lowering operators, $b^s_{\mathbf{p}}, c^s_{\mathbf{p}}$, etc.). Make sure you discard infinite irrelevant additive constants and argue why you are doing it.
(d) Consider a one particle state,

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}}\, b^{s\dagger}_{\mathbf{p}}|0\rangle,$$

and show that this is indeed a state of definite momentum.

Solution:

(a) Under translation $\delta\psi = a^\mu \partial_\mu \psi$, then it is easy to show that, for $\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi$, $\delta\mathcal{L}_D = a^\nu(\delta^\mu_\nu \mathcal{L}_D)$.

(b) Applying Eq. (2.17) in Peskin and Schroeder gives

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu \partial^\nu \psi - g^{\mu\nu}\mathcal{L}_D.$$

However, since we used the equation of motion in deriving the Noether current, and $\mathcal{L}_D = 0$ when the Dirac equation is satisfied, we could just write $T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi$.

(c) The momentum operator $P^i = T^{0i} = i\psi^\dagger \partial^i \psi = -\psi^\dagger \partial_i \psi$, from which we derive

$$\mathbf{P} = \int d^3x \psi^\dagger(-i\nabla)\psi = \sum_r \int \frac{d^3k}{(2\pi)^3} \mathbf{k} \left(b^{r\dagger}_{\mathbf{k}} b^r_{\mathbf{k}} + c^{r\dagger}_{\mathbf{k}} c^r_{\mathbf{k}} \right).$$

(d) Using (c) and the anticommutation relations for the creation and annihilation operators it is easy to show that $\mathbf{P}|\mathbf{p}, s\rangle = \mathbf{p}|\mathbf{p}, s\rangle$.

Problem 4

(a) Derive Eqs. (3.114) and (3.115) in Peskin and Schroeder. Use them to show that $\psi_a(x)$ and $\bar{\psi}_b(y)$ anti-commute at space-like separations. That is, show that $\{\psi_a(x), \bar{\psi}_b(y)\} = 0$ for $(x-y)^2 < 0$.

(b) Consider two operators, $\mathcal{O}_1(x) = \bar{\psi}_a(x) A_{ab} \psi_b(x)$ and $\mathcal{O}_2(x) = \bar{\psi}_a(x) B_{ab}\psi_b(x)$, where A and B are some matrices. (All operators corresponding to physically observable quantities of a fermion field have this generic form – consider for example the energy-momentum tensor and the Noether current encountered earlier in this homework.) Prove that $\mathcal{O}_1$ and $\mathcal{O}_2$ *commute* at space-like separations:

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \quad \text{at} \quad (x-y)^2 < 0,$$

as required by causality.

Solution:

(a) Using the completeness relations for the spinors it is straightforward to prove Eqs. (3.114) and (3.115), which then imply $\{\psi_a(x), \bar{\psi}_b(y)\} = (i\partial\!\!\!/_x + m)_{ab}(D(x-y) - D(y-x))$. We proved in the case of a scalar field that $[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0$ for $(x-y)^2 < 0$.

(b) Using the identity for commutators $[AC, B] = A\{C, B\} - \{A, B\}C$, as well as the result in (a), one can prove the assertion easily.