Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html

SOLUTIONS FOR ASSIGNMENT #7

Reading Assignments:
Sections 4.1 and 4.2 of Peskin and Schroeder.

Problem 1
(a) Show that the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

is invariant under C, P, and T separately.
(Hint: for the charge conjugation, if you blindly use the table in the end of Chapter 3 in Peskin and Schroeder, you will not get the right answer. Instead, start from the basics.)
(b) Replace the derivative ∂^μ by the covariant derivative $D^\mu = \partial^\mu - igA^\mu$ and show that, if electromagnetism is invariant under charge conjugation, the photon A^μ must be odd under charge conjugation

$$CA^\mu(x)C = -A^\mu(x).$$

Solution:
(a) Using the table in the end of Chapter 3 in Peskin and Schroeder, as well as the fact that T-operator is anti-unitary, it is simple to verify that the Dirac Lagrangian is invariant under P and T. For the C operator, we need to work out

$$C(\bar{\psi}\gamma^\mu\partial_\mu\psi)C = (-i\gamma^0\gamma^2\psi)^T\gamma^\mu\partial_\mu(-i\bar{\psi}\gamma^0\gamma^2)^T = -(\partial_\mu\bar{\psi})\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\psi.$$

With the identity for the Pauli matrices $\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^T$ and the chiral representation of Dirac gamma matrices, one sees that $\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2 = \gamma^\mu$. Thus the Dirac Lagrangian is also invariant under C.
(b) With the covariant derivative the Dirac Lagrangian becomes

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu(\partial_\mu - igA_\mu)\psi.$$

Since $\bar{\psi}\partial\!\!\!/\psi$ is even under C and $\bar{\psi}\gamma^\mu\psi$ is odd under C, the photon must be odd under charge conjugation:$CA^\mu C = -A^\mu$ so that $\mathcal{L}_D$ is invariant under C.

Problem 2
Do Problem 3.8 in Peskin and Schroeder.
Solution:
(a) Let's use $(b_k^\pm)^\dagger$ and $(c_k^\pm)^\dagger$ as the creation operator for an electron and a positron, respectively, with spin $\pm 1/2$. A bound state of an electron and a positron is called the positronium.

A total-spin eigenstate of the positronium is created by the operator O^{S,S_z}:

$$O^{1,+1} = (b^+_{k_1})^\dagger (c^+_{k_2})^\dagger, , \quad O^{1,-1} = (b^-_{k_1})^\dagger (c^-_{k_2})^\dagger,$$
$$O^{1,0} = \frac{1}{\sqrt{2}} \left[(b^+_{k_1})^\dagger (c^-_{k_2})^\dagger + (b^-_{k_1})^\dagger (c^+_{k_2})^\dagger \right],$$
$$O^{0,0} = \frac{1}{\sqrt{2}} \left[(b^+_{k_1})^\dagger (c^-_{k_2})^\dagger - (b^-_{k_1})^\dagger (c^+_{k_2})^\dagger \right].$$

In the centre-of-mass frame, $k_1 = -k_2 \equiv k$, then a positronium with orbital angular momentum L and spin S can be written as

$$|L, S, S_z\rangle = \int d^3k \, \psi_L(k) O^{S,S_z} |0\rangle$$

Under parity, $k \to -k$, $b^\dagger_k \to b^\dagger_{-k}$, and $\psi(k) \to \psi(-k) = (-1)^L \psi(k)$. Then under parity $P|L, S, S_z\rangle = (+1)^{L+1} |L, S, S_z\rangle$. The extra (-1) arises from the fact that the intrinsic parity of b_k and c_k is opposite. On the other hand, under charge conjugation $b^\dagger_k \to c^\dagger_k$, which implies $\psi(k) \to \psi(-k) = (-1)^L \psi(k)$ and $O^{S,S_z} \to (-1)^S O^{S,S_z}$. The $(-1)^S$ comes about because 1) the triplet state is symmetric in the spin configuration and the singlet anti-symmetric, and 2) the creation operators anti-commute. In the end, $C|L, S, S_z\rangle = (-1)^{L+S} |L, S, S_z\rangle$.
(b) In the ground state $L = 0$, therefore the $S = 0$ ground state is even under C and can decay into two photons which is also C even. For $S = 1$ ground, it is C odd and must decay through three photons. Similarly, for higher positronium states ${}^{2S+1}L_J$, it can decay into two photons if $L + S$ is even and three photons in $L + S$ is odd.

Problem 3
Consider an interacting theory of four scalar fields $\phi^a, a = 1, 2, 3, 4$ with different masses

$$\mathcal{L}_0 = \sum_{a=1}^{4} \left[\frac{1}{2} (\partial_\mu \phi^a)^2 - \frac{m_a^2}{2} (\phi^a)^2 \right] + \frac{g}{4!} \epsilon_{abcd} \; \epsilon^{\mu\nu\lambda\sigma} \; \partial_\mu \phi^a \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\sigma \phi^d$$

(a) If we define the operation of parity as $P_s : (t, \vec{x}) \to (t, -\vec{x})$ and $P_s : \phi^a(t, \vec{x}) \to \phi^a(t, -\vec{x})$, show that the Lagrangian is parity-violating.
(b) The free part of the Lagrangian has many discrete internal symmetries. One example is $\phi^a(t, -\vec{x}) \to -\phi^a(t, \vec{x})$ for $a = 1, 2, 3, 4$.. Enumerate all possible internal discrete symmetries.
(c) Show that it is possible to define the parity transformation as $P \equiv P_s P_i$, where P_i is one of the internal discrete symmetries, such that the theory is invariant under the newly defined parity P. Do you get $P^2 = 1$? How many pseudo-scalars are in this "parity-invariant" theory?
(d) Now add to the Lagrangian an additional interaction term

$$\mathcal{L}_1 = -g_3 \sum_{a=1}^{4} (\phi^a)^3$$

Show that no definition of parity leaves the Lagrangian invariant. The theory $\mathcal{L}_0 + \mathcal{L}_1$ now violates parity.

(e) Now coonsider a complex scalar χ interacting with ϕ^a through the Lagrangian

$$\mathcal{L} = \sum_{a=1}^{4}\left[\frac{1}{2}(\partial_\mu\phi^a)^2 - \frac{m_a^2}{2}(\phi^a)^2\right] + \partial_\mu\chi^*\partial^\mu\chi - m^2\chi^*\chi - g_3\sum_{a=1}^{4}(\phi^a)^3$$
$$+\frac{g}{4!}\epsilon_{abcd}\,\epsilon^{\mu\nu\lambda\sigma}\,\partial_\mu\phi^a\partial_\nu\phi^b\partial_\lambda\phi^c\partial_\sigma\phi^d(\chi^2+\chi^{*2})$$

Show that there are ways to define the parity P such that $\mathcal{L}$ is parity-invariant. Furthermore, show that you do not get $P^2 = 1$ when the parity acts on some of the scalars in the theory.
Solution:
(a) Because $\epsilon^{\mu\nu\lambda\sigma}$ is odd under P_s, the interaction term breaks parity.
(b) Because the free part of the Lagrangian is quadratic in the field, all possible combinations of $\phi^a \to (\pm 1)\phi^a$ is a symmetry of the free Lagrangian.
(c) Since the interaction term breaks P_s, if it is also odd under P_i, then the interaction term will be invariant. In other words, we need an odd number of pseudo-scalars in the theory in order to preserve parity.
(d) Now an odd number of pseudo-scalars breaks the parity in $\mathcal{L}_1$. This whole Lagrangian is now parity-violating.
(e) If we choose all ϕ^a to be even under parity and the following property for χ

$$P:\chi \to \pm i\chi, \quad P:\chi^* \to \mp i\chi^*,$$

then the theory is parity-invariant. Moreover, $P^2 = -1$ for the χ scalar.