Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

SOLUTIONS FOR ASSIGNMENT #8

Reading Assignments:
Sections 4.3, 4.4, 4.6, and 4.7 of Peskin and Schroeder.

Problem 1
Consider the scalar Yukawa theory we discussed in class

$$\mathcal{L} = \int d^4x\ \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi) - \frac{1}{2}m_\phi^2\phi^2 + \partial_\mu\psi^\dagger\partial^\mu\psi - m_\psi^2\psi^\dagger\psi - g\psi^\dagger\psi\phi$$

In the interaction picture, the quantum field is written in the same way as the free field

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right), \quad \psi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(b_k e^{-ik\cdot x} + c_k^\dagger e^{ik\cdot x}\right).$$

Furthermore, define the amplitude $\mathcal{A}$ of a particular process as

$$\langle f|S-1|i\rangle = i\mathcal{A}\ (2\pi)^4\delta^{(4)}\left(\sum_i k_i - \sum_f k_f\right).$$

(a) Derive the following Wick contractions for the complex scalar ψ:

$$\contraction{}{\psi^\dagger}{(x)}{\psi^\dagger}\psi^\dagger(x)\psi^\dagger(y) = 0, \quad \contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = 0, \quad \contraction{}{\psi^\dagger}{(x)}{\psi}\psi^\dagger(x)\psi(y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m_\psi^2 + i\epsilon}e^{-ik\cdot(x-y)}.$$

(b) Compute $i\mathcal{A}$ in the centre-of-mass frame for the scattering process $\psi(k_1) + \psi^\dagger(k_2) \to \psi(p_1) + \psi^\dagger(p_2)$ using Dyson's formula. (Do not use any Feynman rules here!)
(c) Draw the relevant Feynman diagram(s) for the process in (b). Then use Feynman rules to re-derive your answer in (b).
(d) Compute $i\mathcal{A}$ in the centre-of-mass frame for the scattering process $\psi(k_1) + \phi(k_2) \to \psi(p_1) + \phi(p_2)$ using Dyson's formula. (Do not use any Feynman rules here!)
(e) Draw the relevant Feynman diagram(s) for the process in (d). Then use Feynman rules to re-derive your answer in (d).
(f) Compute $i\mathcal{A}$ in the centre-of-mass frame for the scattering process $\psi(k_1) + \psi^\dagger(k_2) \to \phi(p_1) + \phi(p_2)$ using Dyson's formula. (Do not use any Feynman rules here!)
(g) Draw the relevant Feynman diagram(s) for the process in (f). Then use Feynman rules to re-derive your answer in (f).
(h) Assuming $m_\psi < m_\phi$, what is the minimal velocity of ψ in the centre-of-mass frame in order for the process in (f) to occur?
(i) Compute $i\mathcal{A}$ in the centre-of-mass frame for the decay process $\phi(k) \to \psi(p_1) + \psi^\dagger(p_2)$

using Dyson's formula. (Do not use any Feynman rules here!)
(j) Draw the relevant Feynman diagram(s) for the process in (i). Then use Feynman rules to re-derive your answer in (i).
(k) Can the process in (i) occur for arbitrary masses m_ϕ and m_ψ?
(l) Can the process $\psi(k_1) + \psi(k_2) \to \psi^\dagger(p_1) + \psi^\dagger(p_2)$ occur? Why or why not?
(m) Compute $i\mathcal{A}$ for the scattering process $\phi(k_1) + \phi(k_2) \to \phi(p_1) + \phi(p_2)$ using Dyson's formula. You don't have to evaluate any leftover d^4k integral in the amplitude. (Do not use any Feynman rules here!)
(n) Draw the relevant Feynman diagram(s) for the process in (m). Then use Feynman rules to re-derive your answer in (m).
Solution:
(a) Using

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = \langle 0|T(\psi(x)\psi(y))|0\rangle$$

as well as the commutation relations (of either the fields or the creation/annihilation operators) it is easy to prove the desired.
(b) The amplitude is

$$i\mathcal{A} = -ig^2 \left[\frac{1}{(k_1 + k_1)^2 - m_\phi^2 + i\epsilon} + \frac{1}{(p_1 - k_1)^2 - m_\phi^2 + i\epsilon} \right].$$

In the C.M. frame $k_1 = (E_{c.m.}/2, \vec{k})$, $k_2 = (E_{c.m.}/2, -\vec{k})$, $p_1 = (E_{c.m.}/2, \vec{p})$, and $p_2 = (E_{c.m.}/2, -\vec{p})$, where $E_{c.m.} = m_\psi^2 + |\vec{k}|^2$. Energy conservation requires $|\vec{k}| = |\vec{p}|$. Also let us define $\cos\theta = \vec{k}\cdot\vec{p}/(|\vec{k}||\vec{p}|)$. Then $(k_1 + k_2)^2 = E_{c.m.}^2$ and $(p_1 - k_1)^2 = -2|\vec{k}|^2(1 - \cos\theta)$. Therefore

$$i\mathcal{A} = -ig^2 \left[\frac{1}{E_{c.m.}^2 - m_\phi^2} - \frac{1}{2|\vec{k}|^2(1 - \cos\theta) + m_\phi^2 + i\epsilon} \right].$$

(c) You should draw a t-channel and a s-channel diagram.
(d) The amplitude is

$$i\mathcal{A} = -ig^2 \left[\frac{1}{(k_1 + k_1)^2 - m_\psi^2 + i\epsilon} + \frac{1}{(p_2 - k_1)^2 - m_\psi^2 + i\epsilon} \right].$$

In this case, $|\vec{k}| = |\vec{p}|$, $(k_1 + k_2)^2 = E_{c.m.}^2 = \left(\sqrt{m_\phi^2 + |\vec{k}|^2} + \sqrt{m_\psi^2 + |\vec{k}|^2} \right)^2$, and $(p_2 - k_1)^2 = m_\phi^2 + m_\psi^2 - 2|\vec{k}|^2\cos\theta - 2\sqrt{(m_\phi^2 + |\vec{k}|^2)(m_\psi^2 + |\vec{k}|^2)}$.
(e) Same as (c).
(f) The amplitude is

$$i\mathcal{A} = -ig^2 \left[\frac{1}{(p_1 - k_1)^2 - m_\phi^2 + i\epsilon} + \frac{1}{(p_2 - k_1)^2 - m_\phi^2 + i\epsilon} \right].$$

Here the constraint on the three-momenta is $E_{c.m.}/2 = m_\psi^2 + |\vec{k}|^2 = m_\phi^2 + |\vec{p}|^2$. Then $(p_1 - k_1)^2 = -|\vec{p} - \vec{k}|^2$ and $(p_1 - k_1)^2 = -|\vec{p} + \vec{k}|^2$.

(g) You should draw a t-channel and a u-channel diagram.
(h) Energy conservation requires $m_\psi^2 + |\vec{k}|^2 = m_\phi^2 + |\vec{p}|^2$, so the minimal velocity must satisfy

$$\frac{m_\psi^2 v_{min}^2}{1 - v_{min}^2} + m_\psi^2 \geq m_\phi^2,$$

which implies $v_{min} \geq \sqrt{1 - m_\psi^2/m_\phi^2}$.
(i) The amplitude is $i\mathcal{A} = -ig$.
(j) Same as (i).
(k) We must have $m_\phi \geq 2m_\psi$ in order to conserve energy and momentum.
(l) This process cannot occur because the interaction vertex $\psi^\dagger\psi\phi$ conserves a $U(1)$ global symmetry under which ψ carries $+1$ charge and $\psi^\dagger$ carries -1 charge. The process $\psi(k_1) + \psi(k_2) \to \psi^\dagger(p_1) + \psi^\dagger(p_2)$ violates this global symmetry and cannot occur at *any order* in perturbation theory.
(m) We need to work out the number of *inequivalent* contractions in the time-ordered product

$$\frac{(ig)^4}{4!}\langle\phi(p_1), \phi(p_2)|T(\psi_1^\dagger\psi_1\phi_1\ \psi_2^\dagger\psi_2\phi_2\ \psi_3^\dagger\psi_3\phi_3\ \psi_4^\dagger\psi_4\phi_4)|\phi(k_1), \phi(k_2)\rangle.$$

There are 4! ways to contract the ϕ-field in the four different interaction vertices with the four different external particles, which will cancel the 4! coming from expanding the time-ordered Dyson series up to the fourth order. After the ϕ contraction, we need to contract $\psi^\dagger$ with ψ. Let's assume for now the contraction with the external particles is fixed, *i.e.* each ϕ field is contracted with an external particle, then one example of non-trivial contractions of the ψ field is

$$(1234) \equiv \psi_1^\dagger\psi_1\psi_2^\dagger\psi_2\psi_3^\dagger\psi_3\psi_4^\dagger\psi_4.$$

There are a total of six distinct ways to contract the ψ fields which would give non-trivial contributions to the scattering process. They are $(1234), (1243), (1324), (1342), (1423)$, and (1432), resulting in three different amplitudes

$$\begin{aligned} i\mathcal{A} &= i2g^4 \int \frac{d^4k}{(2\pi)^2}\frac{1}{(k^2 - m_\psi^2 + i\epsilon)((k+k_1)^2 - m_\psi^2 + i\epsilon)} \\ &\quad \times \left[\frac{1}{((k+k_1+k_2)^2 - m_\psi^2 + i\epsilon)((k+p_1)^2 - m_\psi^2 + i\epsilon)}\right. \\ &\quad + \frac{1}{((k+k_1+k_2)^2 - m_\psi^2 + i\epsilon)((k+p_2)^2 - m_\psi^2 + i\epsilon)} \\ &\quad \left. + \frac{1}{((k+k_1-p_2)^2 - m_\psi^2 + i\epsilon)((k+p_2)^2 - m_\psi^2 + i\epsilon)}\right]. \end{aligned}$$

Another way to count these six contractions is from the fact that all four external particles are identical, and they are grouped into two groups of incoming and outgoing particles. So there are $4!/(2!2!) = 6$ different permutations.
(n) It is straightforward to draw these six diagrams.