Northwestern University Department of Physics and Astronomy

PHYSICS 428-2 QUANTUM FIELD THEORY II

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFTII/QFTII_Winter09.html`

THE FINAL EXAM

Due at 3 PM, March 19th

Guidelines:

- There are two problems in this exam. You should try to go as far as you can on both problems. Problem 2 seems long, but don't panic – that's only because I add a lot of explanation and hints.
- The only reference you are allowed to use is the textbook, as well as the homework assignments and the solutions. You must work on the exam by yourself. No discussion is allowed with any other person.
- You are *allowed* to use computer software or calculator for algebraic and numerical computations.
- You are required to derive every equation you write down, except for those derived in the textbook. If you make use of an equation in the textbook, you must cite the original equation in the textbook explicitly. Failure to give the original reference will result in partial or no credit at all.
- It is important for you to clearly state the logic of your answers. I will not make any attempt to "guess" your results. If I cannot follow what you write, I cannot give you the credit.
- Please return your answers to Grant Darktower in the departmental office by the deadline. Or, if you wish, email an electronic version of your answers to me by the deadline.

Problem 1: Renormalization of the Scalar ϕ^3 Theory

In this problem we will consider the scalar ϕ^3 theory in d dimensional spacetime with the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}g\,\phi^3.$$

(a) There is an obvious pathology (instability) in this theory even at the classical level. What is it?

As a toy model, let's ignore the pathology and study its perturbation in quantum theory, which is well-defined.

(b) Determine all superficially divergent Green's functions of $\mathcal{L}$ in $d = 4$ and the corresponding degree of divergence.

(c) Compute all the superficially divergent 1-PI diagrams to one-loop order using Dimensional Regularization (DR). Determine the form of the counter terms necessary to remove the UV divergences. Proceed to renormalize in MS scheme.

(d) Repeat (b) and (c) in $d = 6$.

(e) Compute and renormalize (in MS scheme) the scalar self-energy in $d = 8$. In this case you will need to introduce a counter term corresponding to a higher dimensional operator not already present in $\mathcal{L}$. Write down this operator.

(f) Use dimensional analysis to determine whether the ϕ^3 operator is a relevant, marginal, or irrelevant operator in d =4, 6, and 8, respectively.

(g) Now stay in $d = 4$ and compute the beta function $\beta(g) = (\mu d/d\mu)g(\mu)$, from which derive the effective coupling constant $g(\mu)$. Discuss the asymptotic behavior of the coupling in the UV and the IR, respectively. Does the perturbation theory become more reliable in the UV or in the IR? Including the quantum corrections, is ϕ^3 a relevant, marginal, or irrelevant operator?

Problem 2: Decoupling of Heavy Fermions in QED
The QED Lagrangian with a massive fermion is given by the following Lagrangian:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{D} - M_\Psi)\Psi$$

where the covariant derivative is $D_\mu = \partial_\mu - ieA_\mu$.
(a) Compute the photon self-energy $\Pi(q^2)$ as defined in the textbook to leading order in $\alpha = e^2/4\pi^2$. Use DR to regulate the Feynman amplitude in $d = 4 - 2\epsilon$ and express your answer as an integral over the Feynman parameter.
(b) Renormalize in a momentum-dependent scheme at an Euclidean subtraction point $q^2 = -M^2$. Compute the renormalized $\Pi_r(q^2)$ and Z_3.
(c) Compute the massive fermion contribution to the QED beta function $\beta(e) = (M\partial/\partial M)e$. Again express your answer as an integral over the Feynman parameter. Verify that when the fermion is light comparing to the subtraction point, $M_\Psi \ll M$, $\beta(e)$ reduces to the case of QED with a massless fermion (as computed in Eq. (12.61) in the textbook.) On the other hand, when the fermion becomes heavy, $M_\Psi \gg M$, its contribution to $\beta(e)$ and $\Pi_r(q^2)$ vanishes as

$$\lim_{M_\Psi \gg M} \beta(e) = C_1 \frac{M^2}{M_\Psi^2}, \quad \Pi_r(q^2) = C_2 \frac{q^2 + M^2}{M_\Psi^2}.$$

Compute the constants C_1 and C_2.

The moral of the above story is that a massive fermion decouples from low-energy physics when its mass becomes much heavier than the energy scale we do the measurements. Very intuitive, right? Well, until you use a momentum-independent renormalization scheme.

(d) Compute $\Pi_r(q^2)$ and Z_3 in the minimal-subtraction (MS) scheme. Compute $\beta(e)$ in MS scheme and show that it is independent of the mass M_Ψ of the heavy fermion and the renormalization scale μ. In the limit $M_\Psi \gg \mu$, determine from the form of $\Pi_r(q^2)$ in MS scheme whether you can trust the perturbation theory or not.

So it seems that the heavy fermion do not decouple in the MS scheme as it should. This is an artifact of using a non-decoupling scheme like the

MS scheme. The decoupling in MS-like schemes can be achieved by explicitly constructing an effective field theory, valid below the mass of the heavy fermion, where the heavy particle is "integrated out." To see how this works, let's consider an effective Lagrangian with only the photon:

$$\mathcal{L}_{eff} = -\frac{1}{2}(\mathcal{F}_{\mu\nu})^2,$$

where $\mathcal{A}_\mu$ is the effective photon in the effective field theory, and may not be identical to the corresponding A_μ in the full QED with a heavy fermion.

The requirement of a valid effective field theory is such that $\mathcal{L}_{eff}$ reproduces exactly the same low-energy physics as in $\mathcal{L}_{full}$. In other words, we "match" the Green's functions in $\mathcal{L}_{eff}$, where the external legs are the light states $\mathcal{A}$, to the corresponding Green's functions in the full theory with the corresponding A as the external legs, at a scale close to M_Ψ. (Such a procedure is called "matching.")

(e) Compute the renormalized two-point function in the full QED Lagrangian $\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle$ by calculating the renormalized self-energy $\Pi_r(q^2)$ to order α. Match the result to the two-point function in the effective theory $\langle 0|T(\mathcal{A}_\mu(x)\mathcal{A}_\nu(y))|0\rangle$ and show that, in the limit $|q^2| \ll M_\Psi^2$, one must have

$$\mathcal{A}_\mu(x) = (1+\Pi_r(0))^{\frac{1}{2}} A_\mu(x).$$

At order α the self-energy $\Pi_r(q^2)$ depends on the renormalization scale μ. Explain why you should choose $\mu \approx M_\Psi$.

(f) Now consider adding a light fermion ψ with mass $m_\psi \ll M_\Psi$ to both the full QED and the effective Lagrangians:

$$\begin{aligned}\mathcal{L}^{\psi}_{full} &= \bar{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m_\psi)\psi,\\ \mathcal{L}^{\psi}_{eff} &= \bar{\psi}(i\partial\!\!\!/ + e_{eff}\mathcal{A}\!\!\!/ - m_\psi)\psi,\end{aligned}$$

where e_{eff} is the electric charge in the effective field theory. Use gauge invariance to argue that

$$e_{eff} = (1+\Pi_r(0))^{-\frac{1}{2}} e.$$

Moreover, show that the electric charge is the same at the matching scale $\mu = M_\Psi$: $e_{eff}(M_\Psi) = e(M_\Psi)$.

(g) Notice that the renormalized effective coupling $e_{eff} = e_{eff}(e;\mu, M_\Psi)$, which implies the renormalization group evolution of e_{eff} is given by

$$\mu\frac{d}{d\mu}e_{eff} = \left(\mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + \gamma_{M_\Psi} M_\Psi \frac{\partial}{\partial M_\Psi}\right) e_{eff} \equiv \beta_{eff}(e_{eff}),$$

where $\beta(e)$ and γ_{M_Ψ} are the beta function and the anomalous dimension of the heavy mass parameter M_Ψ in the *full* QED Lagrangian with one heavy and one light fermions. Compute effective beta function $\beta_{eff}(e_{eff})$ in the effective theory and show that the heavy fermion is now decoupled and does not contribute, contrary to the situation in (d).

To summarize, the way the decoupling works in MS-like scheme is to explicitly construct an effective field theory where a heavy particle is removed by hand. One then matches the effective theory to the full theory at the scale $\mu \approx M_\Psi$. We have seen that the coupling constant is continuous at the matching scale $\mu = M_\Psi$ in (f). (This is true only at one-loop though.) However, the beta function is discontinuous as one moves across the heavy particle threshold M_Ψ: above the threshold the beta receives contributions from both the heavy and light particles, while below the threshold it receives contributions only from the light particles, as demonstrated in (g).