QFT II, Winter 2009.

HW #10, Solutions.

Problem 1 (P&S 9.2).

(a) The trace of an operator on the Hilbert space is

$$\mathrm{Tr}\, \mathcal{O} = \int dx \, \langle x | \mathcal{O} | x \rangle \quad \longrightarrow \text{position eigenstate.}$$

For $\mathcal{O} = e^{-\beta H}$, we can divide up $\mathcal{O}$ into small pieces

$$\mathrm{Tr}\, e^{-\beta H} = \int dx \, \langle x | e^{-\beta H} | x \rangle$$

$$= \int dx \prod_{i=1}^{N-1} \langle x | e^{-\beta H/N} | x_{N-1} \rangle \, dx_{N-1} \langle x_{N-1} | e^{-\beta H/N} | x_{N-2} \rangle$$

$$\times \, dx_{N-2} \langle x_{N-2} | \cdots \quad | x_1 \rangle \, dx_1 \langle x_1 | e^{-\beta H/N} | x \rangle$$

For the Hamiltonian $H = \frac{p^2}{2m} + V(x)$, and a small number $\varepsilon = \beta/N$.

$$\langle x | e^{-\varepsilon H} | y \rangle = \int \langle x | e^{-\varepsilon p^2/2m} | p \rangle dp \langle p | e^{-\varepsilon V(x)} e^{-O(\varepsilon^2)} | y \rangle$$

$$= \frac{1}{2\pi\hbar} \sqrt{\frac{2\pi m}{\varepsilon}} \, e^{-m(x-y)^2/2\varepsilon\hbar^2} \, e^{-\varepsilon V(y)}$$

Therefore, using $x_0 = x_N = x$

$$\mathrm{Tr}(e^{-\beta H}) = \int \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi\hbar^2\varepsilon}}\, e^{-S_E/\hbar}$$

where
$$S_E = \hbar \cdot \sum_{i=1}^{N} \left[\frac{m(x_i - x_{i-1})^2}{2\varepsilon\hbar^2} + \varepsilon V(x_i) \right]$$

$$= \frac{\hbar\beta}{N} \sum_{i=1}^{N} \left[\frac{m(x_i - x_{i-1})^2}{2(\hbar\beta/N)^2} + V(x_i) \right]$$

$$\to \oint_{\substack{\Delta\tau = \hbar\beta \\ x(\hbar\beta) = x(0).}} \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right] d\tau.$$

This is nothing but the Euclidean action. ($\tau = it$!)

(b) $L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$. Set $\hbar = 1$. below

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}.$$

$x(t)^* = x(t) \Rightarrow x_{-n} = x_n^*$

x_0 is real.

$$S_E = \oint_{\Delta\tau = \beta} d\tau \sum_{n,m} \frac{1}{2}\left[\frac{2\pi i n}{\beta} \frac{2\pi i m}{\beta} + \omega^2 \right] \frac{1}{\beta} x_n e^{2\pi i n \tau/\beta} x_m e^{2\pi i m \tau/\beta}$$

$$= \frac{1}{2}\omega^2 x_0^2 + \sum_{n=1}^{\infty} \left[\frac{(2\pi n)^2}{\beta^2} + \omega^2 \right] |x_n|^2.$$

$$\int d\tau\, e^{2\pi i n \tau/\beta} e^{2\pi i m \tau/\beta} \sim \delta_{m,-n}$$

The path integral is Gaussian and given by product of eigenvalues up to a β-dependent factor

$$Z \propto \left[\omega \prod_{n=1}^{\infty} \left[\frac{(2\pi n)^2}{\beta^2} + \omega^2 \right] \right]^{-1} \propto \left[\frac{\beta\omega}{2} \prod_{n=1}^{\infty} \left(1 + \frac{\beta^2\omega^2}{(2\pi n)^2} \right) \right]^{-1} = \left[\sinh \frac{\beta\omega}{2} \right]^{-1}$$

where we've used the identity given in the hint in P&S.

$$\therefore \quad Z \propto \frac{1}{e^{\frac{\beta\omega}{2}} - e^{-\beta\omega/2}} = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

, which matches the partition function of a SHO.

$$Z_{SHO} = \sum_{n=0}^{\infty} e^{-\beta\omega(n+\frac{1}{2})} = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

Rewrite $S_E = \oint_{0\tau=\beta} d\tau \; \frac{1}{2} x(\tau) \left[-\partial_\tau^2 + \omega^2\right] x(\tau)$.

$$\Rightarrow \quad Z = \left[\det(-\partial_\tau^2 + \omega^2)\right]^{-1/2} = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}.$$

c) In field theory.

$$Z = \int \mathcal{D}\phi(\vec{x},\tau) \; e^{-S_E}$$

$$S_E = \int d\vec{x} \oint_0^\beta d\tau \; \frac{1}{2} \phi(-\partial^2 + m^2)\phi.$$

$$\therefore Z = \left[\det(-\partial^2 + m^2)\right]^{-1/2} = \left[\det(-\partial_\tau^2 - \vec{\nabla}^2 + m^2)\right]^{-1/2}$$

$$= \prod_{\vec{p}} \left[\det(-\partial_\tau^2 + \underbrace{\vec{p}^2 + m^2}_{\equiv \omega(\vec{p})!})\right]^{-1/2}$$

results from (b).

$$= \prod_{\vec{p}} \frac{e^{-\beta\omega(\vec{p})/2}}{1-e^{-\beta\omega(\vec{p})}}$$

This is indeed the partition function of a relativistic boson of mass m.

(d).

$$S_E = \oint_{0<\tau<\beta} d\tau \left[\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi\right] \qquad \dot{\psi} = \frac{\partial}{\partial\tau}\psi$$

The anti-periodic b.c. $\Rightarrow$

$$\psi(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \psi_n \, e^{2\pi i(n+\frac{1}{2})\tau/\beta}$$

$$\bar{\psi}(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \bar{\psi}_n \, e^{-2\pi i(n+\frac{1}{2})\tau/\beta}.$$

$$\Rightarrow S_E = \oint d\tau \frac{1}{\beta} \sum_{m,n} \bar{\psi}_n e^{-2\pi i(n+\frac{1}{2})\tau/\beta} \left[\frac{2\pi i}{\beta}\left(m+\frac{1}{2}\right) + \omega\right] \psi_m e^{2\pi i(m+\frac{1}{2})\tau/\beta}$$

$$= \sum_n \bar{\psi}_n \left[\frac{2\pi i}{\beta}\left(n+\frac{1}{2}\right) + \omega\right] \psi_n.$$

$$\Rightarrow Z = \int \prod_n d\psi_n d\bar{\psi}_n \, e^{-S_E} = \prod_{n=-\infty}^{+\infty} \left[\frac{2\pi i}{\beta}\left(n+\frac{1}{2}\right) + \omega\right]$$

$$= \prod_{n=0}^{\infty} \left[\frac{2\pi i}{\beta}\left(n+\frac{1}{2}\right) + \omega\right]\left[-\frac{2\pi i}{\beta}\left(n+\frac{1}{2}\right) + \omega\right]$$

from 0 to ∞. !

$$= \prod_{n=0}^{\infty} \left[\left(\frac{2\pi}{\beta}\right)^2 \left(n+\frac{1}{2}\right)^2 + \omega^2\right] \qquad \text{use } \cosh x = \prod_{n=0}^{\infty}\left[1 + \frac{x^2}{\pi^2(n+\frac{1}{2})^2}\right]$$

$$\propto \cosh\frac{\beta\omega}{2} \propto \cancel{\cosh} \; e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}}$$

This corresponds to a QM system w/ two energy states $H = +\frac{\omega}{2}, -\frac{\omega}{2}$, by definition of the partition function!

(e).

$$Z = \int \mathcal{D}A_\mu \, e^{-\int d^4x \left(\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2\right)} \left(\det \partial^2\right).$$

In Feynman gauge $\xi = 1$,

$$S_E = \int d^4x \left(\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(\partial_\mu A_\mu)^2\right)$$

$$= -\frac{1}{2}\int d^4x \, A_\mu \partial^2 A_\mu$$

massless!

which is the action for 4 scalar fields ($\mu = 0, 1, 2, 3$), each contributing $(\det \partial^2)^{-1/2}$ to the Z.

$$\Rightarrow \quad Z = \left(\left[\det \partial^2\right]^{-1/2}\right)^4 \cdot \left(\det \partial^2\right) = \left(\det \partial^2\right)^{-1}$$

$$= \left[\left(\det \partial^2\right)^{-1/2}\right]^2.$$

This is the # of polarization in the end, which is exactly 2!

<u>Problem 2</u> It's straightforward to derive the equation using results in Chap 2 of P&S. The delta function comes from the time-ordering of the Green's function.