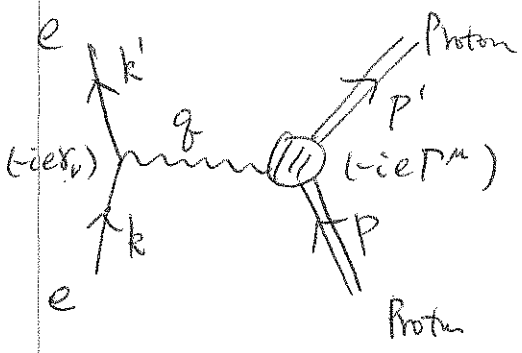
QFT II

HW1 Solutions

Problem 1 (P&S Prob. 6.1).

<u>Solution:</u>



$$i\mathcal{M} = \bar{u}(k')(-ie\gamma_\mu)u(k)\,\frac{-i}{q^2}\,\bar{u}(p')(-ie\Gamma^\mu)u(p)$$

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$$

m is the proton mass and $m_e \to 0$.

<u>STEP 1</u>: use the Gordon identity to write

$$\Gamma^\mu = \gamma^\mu (F_1 + F_2) - \frac{1}{2m}(p+p')^\mu F_2.$$

<u>STEP 2</u>: Square the amplitude by computing the traces

$$|\overline{\mathcal{M}}|^2 = \frac{8e^4}{q^4}\Big[(F_1+F_2)^2\big((k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - m^2 k'\cdot k\big)$$

$$+\Big(k'\cdot(p+p')\,k\cdot(p+p') - \tfrac{1}{2}(p+p')^2(k\cdot k')\Big)\Big(\frac{p'\cdot p+m^2}{4m^2}F_2^2 - F_2(F_1+F_2)\Big)\Big]$$

STEP 3: Kinematics in the LAB frame

$$p = (m, 0) \quad ; \quad k = (E, \vec{k}) \quad , \quad \vec{k} = E\hat{z}$$
$$k' = (E', \vec{k}') \quad , \quad |\vec{k}'| = E'$$

One can show that $\quad E' - E = \dfrac{q^2}{2m}$

and $\quad q^2 = -\dfrac{4E^2 \sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}}$

STEP 4: Compute $|\overline{m}|^2$ in the LAB frame

$$|\overline{m}|^2 = \frac{e^4 m^2}{E^2 \sin^4\frac{\theta}{2}}\left(1 + \frac{2E}{m}\sin^2\frac{\theta}{2}\right)$$
$$\times\left[\cos^2\frac{\theta}{2}\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right) - \sin^2\frac{\theta}{2}\cdot\frac{q^2}{2m^2}(F_1 + F_2)^2\right]$$

then

$$\left(\frac{d\sigma}{d\cos\theta}\right)_{LAB} = \frac{1}{32\pi m^2\left(1 + \frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\,|\overline{m}|^2.$$

Problem 2

Solution :

(a) The two relevant sources of $\hbar$ are

i) Canonical quantization

$$[\varphi, \pi] = i\delta^{(3)} \rightarrow [\varphi, \pi] = i\hbar\delta^{(3)}(\ ).$$

which implies a factor of $\hbar$ in the propagator

$$\overbrace{\varphi\varphi} \sim \frac{i\hbar}{p^2 - m^2 + i\varepsilon}.$$

ii) time-evolution operator.

$$e^{-iHt} \rightarrow e^{-iHt/\hbar}$$

In terms of kinetic energy T and potential energy V

$$H = T + V$$

$\Rightarrow$ each interaction vertex in V gets a factor of $\hbar^{-1}$

Thus for each diagram with I, V, and E, the number of propagators is $I+E$, whereas the number of vertices is V.

$\Rightarrow$ the power of $\hbar$ is $\hbar^{I+E-V} = \hbar^{E-1+L}$

where $L = I-(V-1)$ is the number of loops as shown in class.

(b) In ϕ^4 theory, each vertex has 4 internal or external lines connecting to it. And each internal line connects to two vertices. So we have

$$4V = E + 2I \quad \Rightarrow E \text{ must be even!}$$

Using $L = I-(V-1)$ we get

$$V = L - 1 + E/2 \quad = \text{Power of } \lambda.$$

Problem 3

Solution:

(a)

$$\prod_{m=1}^{n} \frac{1}{A_m+i\varepsilon} = \prod_{m=1}^{n}\left[-i\int_0^\infty d\alpha_m\, e^{i\alpha_m(A_m+i\varepsilon)}\right]$$

$$= (-i)^n \int_0^\infty d\alpha_1 \cdots d\alpha_n\, e^{i\sum_m \alpha_m(A_m+i\varepsilon)} \underbrace{\int_0^\infty d\lambda\, \delta(\lambda-\sum_s \alpha_s)}_{\text{Insert "1" here!}}$$

Now write

$$\delta(\lambda-\sum_s\alpha_s) = \frac{1}{\lambda}\delta(1-\sum_s\tilde{\alpha}_s) \quad \tilde{\alpha}_s = \frac{\alpha_s}{\lambda}$$

$$\Rightarrow (-i)^n \int_0^\infty d\tilde{\alpha}_1 \cdots d\tilde{\alpha}_n\, \delta(1-\sum_s\tilde{\alpha}_s) \underbrace{\int_0^\infty \lambda^{n-1} d\lambda\, e^{i\lambda\sum_m\tilde{\alpha}_m(A_m+i\varepsilon)}}_{= \frac{(n-1)!}{\left[-i\sum_m \tilde{\alpha}_m(A_m+i\varepsilon)\right]^n}}$$

$$\therefore \prod_{m=1}^{n}\frac{1}{A_m+i\varepsilon} = (n-1)!\int_0^1 d\alpha_1\cdots d\alpha_n\, \delta(1-\sum_s\alpha_s)\frac{1}{\left[\sum_m \alpha_m(A_m+i\varepsilon)\right]^n}$$

(b) $$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}$$

$$\xRightarrow{t\to At} \frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-1} e^{-At}$$

Then use the trick in (a) to insert "1".

Problem 4

Solution :

The easiest way to prove this identity is to use the result of Prob. 3.6 of P&S. and write

$$\gamma^\mu \gamma^\nu \gamma^\rho = \sum_A c_A \Gamma^A$$

where $\Gamma^A = \{1, \gamma^0, i\gamma^5, \cdots\}$. $\mathrm{Tr}[\Gamma^A \Gamma^B] = 4\delta^{AB}$

then $c_A = \frac{1}{4} \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \Gamma^A]$

Problem 5

Solution :

I didn't realize this, but this equation has been proved on P.249 of P&S !!