QFT II, Winter '09

Solutions for HW 3

Problem 1

(a) use the identity operator

$$\mathbb{I} = \sum_n |n\rangle\langle n|.$$

then

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi^\dagger(y)|0\rangle$$

$$= \sum_n e^{ip_n\cdot(x-y)} \, |\langle 0|\phi(0)|n\rangle|^2$$

Define $\rho_1(\mu^2)$ in such a way

$$(2\pi)^3 \sum_n \delta^{(4)}(p-p_n)\,|\langle 0|\phi(0)|n\rangle|^2 = \theta(p^0)\,\rho_1(p^2)$$

$$\Rightarrow \langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \cancel{\frac{1}{(2\pi)^3}}\sum_n \int d^4p\; e^{ip\cdot(x-y)}\,\delta^{(4)}(p-p_n)\,|\langle 0|\phi(0)|n\rangle|^2$$

$$= \frac{1}{(2\pi)^3}\int d^4p\; e^{ip\cdot(x-y)}\,\theta(p^0)\,\rho_1(p^2)$$

$$= \frac{1}{(2\pi)^3}\int d^4p \int_0^\infty d\mu^2\; e^{ip\cdot(x-y)}\,\theta(p^0)\,\rho_1(\mu^2)\,\delta(p^2-\mu^2)$$

$$= \int_0^\infty d\mu^2\,\rho_1(\mu^2)\,D(x-y;\mu^2)$$

where

$$D(x-y;\mu^2) = \frac{1}{(2\pi)^3}\int d^4p\; e^{ip\cdot(x-y)}\,\theta(p^0)\,\delta(p^2-\mu^2)$$

For $\langle 0|\phi^\dagger(y)\,\phi(x)|0\rangle$ we can define a similar spectral decomposition

$$\langle 0|\phi^\dagger(y)\,\phi(x)|0\rangle = \int_0^\infty d\mu^2 \rho_2(\mu^2)\, D(y-x;\mu^2)$$

where ~~$\theta(p^0)\,\rho_2(\mu^2) =$~~

$$\theta(p^0)\,\rho_2(p^2) = \sum_n \delta^{(4)}(p-p_n)\,|\langle n|\phi^\dagger(0)|0\rangle|^2$$

Thus
$$\langle 0|[\phi(x),\phi^\dagger(y)]|0\rangle = \int_0^\infty d\mu^2 \Big(\rho_1(\mu^2)\,D(x-y;\mu^2) - \rho_2(\mu^2)\,D(y-x;\mu^2)\Big)$$

Causality requires $\langle 0|[\phi(x),\phi^\dagger(y)]|0\rangle = 0$ for space-like separations $(x-y)^2 < 0$

$$\Rightarrow \quad \rho_1(\mu^2) = \rho_2(\mu^2)$$

then
$$\langle 0|[\phi(x),\phi^\dagger(y)]|0\rangle$$
$$= \int_0^\infty d\mu^2\, \rho(\mu^2)\Big(D(x-y;\mu^2) - D(y-x;\mu^2)\Big)$$

(b).

the canonical quantization is

$\cancel{[\phi^\dagger(y,t)}$

$$[\phi^\dagger(\vec{y},t), \partial_0\phi(\vec{x},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$$

Also from the definition of $D(x-y;\mu^2)$

we see

$$\frac{\partial}{\partial x^0} D(x-y;\mu^2)\Big|_{x^0=y^0} = -i\,\delta^{(3)}(\vec{x}-\vec{y})$$

Then the spectral representation in (a) implies

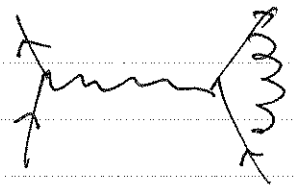
$$\int_0^\infty d\mu^2 \rho(\mu^2) = 1 \qquad \underline{\underline{Z}}$$

Since $\rho(\mu^2) = Z\,\delta(\mu^2-m^2) + \cdots\cdots$

$\Rightarrow \quad Z \leq 1.$

Problem 2. P&S Part I, Final Project.

(a). The diagram is



The amplitude is the same as in Eq. (6.54) in P&S w/ $\alpha \to \alpha_g$.

$$= \frac{\alpha_g}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$

$$\times \bar{u}(p')\left\{\gamma^\mu\left[\log\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}\left((1-x)(1-y)q^2 + (1-4z+z^2)m_q^2\right)\right]\right.$$

$$\left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m_q}\left[\frac{1}{\Delta}2m_q^2 z(1-z)\right]\right\}$$

$$\Delta = -xyq^2 + (1-z)^2 m_q^2.$$

Subtract at $q^2=0$ we have, as in Eq. (6.56), (6.57).

$$\delta F_1(q^2) = \frac{\alpha_g}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left\{\log\frac{m_q^2(1-z)^2+\mu^2 z}{m_q^2(1-z)^2 - q^2xy + \mu^2 z}\right.$$

$$\left. + \frac{m_q^2(1-4z+z^2) + q^2(1-x)(1-y)}{m_q^2(1-z)^2 - q^2xy + \mu^2 z} - \frac{m_q^2(1-4z+z^2)}{m_q^2(1-z)^2+\mu^2 z}\right\}.$$

$$\delta F_2(q^2) = \frac{\alpha_g}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[\frac{2m_q^2(1-z)\cdot z}{m_q^2(1-z)^2 - q^2xy}\right]$$

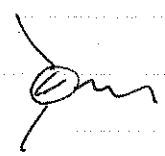
Notice that we have put in a small photon mass, as in Eq. (6.56) + (6.57).

However, a crucial difference is we also put the photon mass term in the logarithm in $F_1(q^2)$, which otherwise is (IR) divergent in the limit $m_q \to 0$.

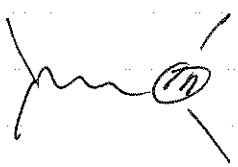
$\delta F_2(q^2)$ on the other hand, vanishes as $m_q \to 0$.

In this limit,

$= \cancel{-i\gamma^\mu Q_f} \; -i\gamma^\mu F_1(q^2) \equiv$



The $e^+e^- \to q\bar{q}$ amplitude is now



$$\therefore \quad \sigma(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3s} 3 \cdot |F_1(q^2=s)|^2$$