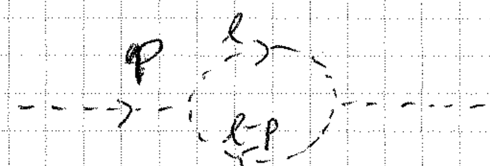
QFT II, Winter 2008.

HW 4
Solutions

Problem 1 $\quad \phi^3$ in $D=4$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{3!}\phi_0^3$$

at 1-loop, $\Sigma(p^2; m)$ has the diagram



$$\Sigma^{(2)}(p^2, m) = (i\lambda)^2 \frac{\mu^{2\varepsilon}}{2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m^2 + i\varepsilon} \frac{1}{(l-p)^2 - m^2 + i\varepsilon}$$

Symmetry factor!

$$= \frac{1}{2}\lambda^2 \mu^{2\varepsilon} \frac{\Gamma(\varepsilon)}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \left[m^2 - x(1-x)p^2 - i\varepsilon\right]^{-\varepsilon}$$

$$= \frac{1}{2}\lambda^2 \frac{1}{16\pi^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + 2 + \log\frac{4\pi\mu^2}{m^2} + \int_0^1 dx \log\frac{1}{1 - x(1-x)\frac{p^2}{m^2}} \right\}.$$

The divergent pole $\frac{1}{\varepsilon}$ only multiplies a constant, so we only need the mass counter term

$$-\frac{1}{2}\delta_m \phi^2$$

but not the $\delta_p p^2$ counter term.

Imposing on-shell condition

$$\Sigma^{(R)}(p^2, m)\Big|_{p^2=m^2} = 0$$

Write the counter term as finite piece

$$\delta_m = -\frac{1}{2}\lambda^2 \frac{1}{16\pi^2}\left(-\frac{1}{\varepsilon} + c_m\right)$$

$$\Sigma^{(R)}(p^2, m) = \frac{1}{2}\lambda^2 \frac{1}{16\pi^2}\left\{\frac{1}{\varepsilon} - \gamma_E + 2 + \log\frac{4\pi\mu^2}{m^2} - \left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}} \log\left[\frac{\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}+1}{\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}-1}\right]\right\} + \delta_m$$

$$= \frac{1}{2}\lambda^2 \frac{1}{16\pi^2}\left\{-\gamma_E + 2 + \log\frac{4\pi\mu^2}{m^2} - \left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}} \times \log\left[\frac{\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}+1}{\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}-1}\right] + c_m\right\}$$

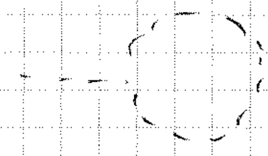
$$\Rightarrow \quad c_m = \gamma_E - 2 - \log\frac{4\pi\mu^2}{m^2} + \underbrace{\frac{\pi}{\sqrt{3}}}$$

$$\Downarrow$$

$$\lim_{p^2\to m^2}\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}\log\left[\frac{\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}+1}{\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}-1}\right] = \frac{\pi}{\sqrt{3}}$$

(b).

The tadpole diagram

 $= i\tau$

$$i\tau = -\frac{1}{2} i \lambda \mu^{\varepsilon} \frac{\Gamma(\varepsilon-1)}{(4\pi)^{2-\varepsilon}} \cdot (m^2)^{1-\varepsilon}$$

a c.t. of the form

$\mathcal{L}_{tad} = -\tau_m \phi(x)$ is induced

The condition that the tadpole vanishes gives $\tau_m = \tau$.

Problem 2

i) $x_i = \dfrac{2 k_i \cdot q}{q^2}$ $\quad q = k_1 + k_2 + k_3 = p_{e^+} + p_{e^-}$

$q^2 = E_{cm}^2$

$$\sum_i x_i = \frac{2 q \cdot q}{q^2} = 2$$

ii) The Lorentz scalars involving final-state momenta are

$k_1 \cdot k_2$, $k_1 \cdot k_3$, $k_2 \cdot k_3$, $k_1^2 = m_q^2 \equiv m_1^2$, $k_2^2 = m_{\bar{q}}^2 \equiv m_2^2$, $k_3^2 = m_g^2 \equiv m_3^2$

then $x_i = \dfrac{2}{q^2}\left(k_i^2 + \sum_{j \neq i} k_i \cdot k_j\right)$.

Note that q^2 is a constant since $q^2 = E_{cm}^2$

then it's easy to solve

$$k_i k_j = \frac{1}{2}(m_i^2 + m_j^2 - m_k^2) + \frac{q^2}{4}(\underbrace{x_i + x_j}_{=2-x_k\,!} - x_k)$$

(iii) $$\int d\Pi_3 = \int \prod_i \frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_i}(2\pi)^4 \delta^{(4)}(q - \sum_i k_i).$$

The integration is Lorentz invariant $\Rightarrow$ choose CM frame

$q = (E_{CM}, \vec{0})$ $\qquad \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$

$\qquad E_3 = \sqrt{\mu^2 + |\vec{k}_1 + \vec{k}_2|^2}$

$$\Rightarrow \int d\Pi_3 = \int \frac{1}{(2\pi)^5} d^3k_1\, d^3k_2 \frac{1}{8E_1E_2E_3} \cdot \delta(E_{CM} - E_1 - E_2 - E_3)$$

Write $d^3k_1 = |\vec{k}_1|^2 d|\vec{k}_1|\, d\Omega_1$

For d^3k_2, write it in terms of the "relative" azimuthal and polar angles w.r.t. $\vec{p}_1$

$$d^3k_2 = |\vec{k}_2|^2 d|\vec{k}_2|\, d\Omega_{12}$$
$$= |\vec{k}_2|^2 d|\vec{k}_2| \cdot d\varphi_{12}\, d\cos\theta_{12}$$

Notice that $\quad E_3^2 = \mu^2 + |\vec{k}_1|^2 + |\vec{k}_2|^2 + 2|\vec{k}_1||\vec{k}_2|\cos\theta_{12}.$

Thus $\quad \frac{\partial E_3}{\partial \cos\theta_{12}} = \frac{|\vec{k}_1||\vec{k}_2|}{E_3}$ and

$$d\cos\theta_{12} \cdot \delta(E_{CM} - E_1 - E_2 - E_3) = \frac{E_3}{|\vec{k}_1||\vec{k}_2|}.$$

$$\Rightarrow \int d\Pi_3 = \int \frac{1}{256\pi^5} \frac{|\vec{k}_1|^2 d|\vec{k}_1|\,|\vec{k}_2|\,d|\vec{k}_2|^2\, d\Omega_1 d\varphi_{12}}{\cancel{8}E_1E_2E_3} \frac{E_3}{|\vec{k}_1||\vec{k}_2|}$$

$$= \int \frac{1}{256\pi^5} \underbrace{\frac{|\vec{k}_1|\,d|\vec{k}_1|}{E_1}}_{=dE_1} \underbrace{\frac{|\vec{k}_2|\,d|\vec{k}_2|}{E_2}}_{=dE_2} d\Omega_1 d\varphi_{12}$$

The $|\text{amplitude}|^2$, or the differential cross-section is independent of the angles if we average over initial spin states.

$\therefore$ $d\Omega_1 d\varphi_{12}$ can be performed to get $8\pi^2$

$$\Rightarrow \int d\Pi_3 = \frac{1}{32\pi^4} \int dE_1 dE_1.$$

In the CM frame

$$x_i = \frac{2k_i \cdot q}{q^2} = \frac{2E_i}{E_{CM}}$$

$$\Rightarrow \int d\Pi_3 = \frac{E_{CM}^2}{128\pi^4} \int dx_1 dx_2 = \frac{q^2}{128\pi^4} \int dx_1 dx_2$$

<u>Kinematic limits</u>:

From $E_3^2 = \mu^2 + |\vec{k}_1|^2 + |\vec{k}_2|^2 + 2|\vec{k}_1||\vec{k}_2| \cos\theta_{12}$

We have $x_3 = \left(\frac{4\mu^2}{q^2} + x_1^2 + x_2^2 + 2x_1x_2\cos\theta_{12} \right)^{\frac{1}{2}}$

then

$$x_1+x_2+x_3=2$$

$$\Rightarrow \quad x_1+x_2+\sqrt{\frac{4\mu^2}{q^2}+x_1^2+x_2^2+2x_1x_2\cos\theta_{12}}=2.$$

or

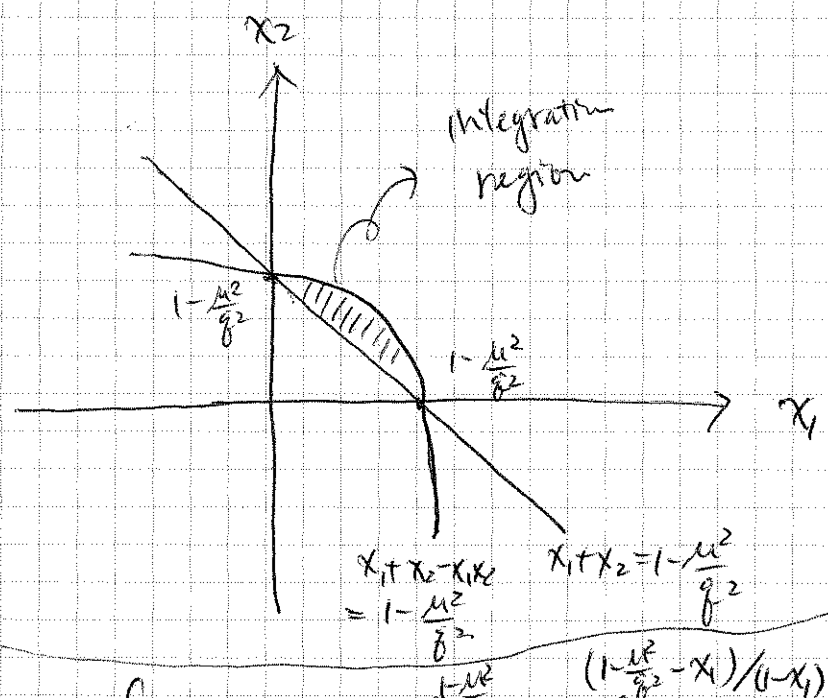
$$x_1+x_2+\frac{1}{2}x_1x_2(\cos\theta_{12}-1)=1-\frac{\mu^2}{q^2}$$

since $-1\le\cos\theta_{12}\le 1$

$$\Rightarrow \quad -1 \le \cos\theta_{12}=\frac{1-\frac{\mu^2}{q^2}-(x_1+x_2)}{\frac{1}{2}x_1x_2}+1 \le 1$$

$$\Rightarrow \quad \begin{cases} x_1+x_2 \ge 1-\dfrac{\mu^2}{q^2} \\ x_1+x_2-x_1x_2 \le 1-\dfrac{\mu^2}{q^2} \end{cases}$$

note $1-\frac{\mu^2}{q^2}>0$ (μ^2 is small)



$$\therefore \int dx_1\,dx_2 = \int_0^{1-\frac{\mu^2}{q^2}} dx_1 \int_{1-\frac{\mu^2}{q^2}-x_1}^{(1-\frac{\mu^2}{q^2}-x_1)/(1-x_1)} dx_2 = \int_0^{1-\frac{\mu^2}{q^2}} dx_2 \int_{1-\frac{\mu^2}{q^2}-x_2}^{\frac{(1-\frac{\mu^2}{q^2}-x_2)}{(1-x_2)}} dx_1$$