QFT II, Winter 2009 (R1)

HW #5 Solutions

Problem 1

(a)
$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+i\varepsilon} = \underbrace{\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-M^2+i\varepsilon}}_{(a)} - \underbrace{\int \frac{d^dk}{(2\pi)^d}\frac{M^2}{(k^2+i\varepsilon)(k^2-M^2+i\varepsilon)}}_{(b)}$$

$$(a) = (-1)\, i\pi^{\frac{d}{2}}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)}(M^2)^{\frac{d}{2}-1}$$

$$(b) = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{M^2}{[k^2-xM^2]^2}$$

$$= \int_0^1 dx\, M^2 \cdot i\pi^{\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}(xM^2)^{\frac{d}{2}-2}$$

$$= \frac{1}{\frac{d}{2}-1}\, i\pi^{\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}(M^2)^{\frac{d}{2}-1}$$

Use the identity $\Gamma(x+1) = x\Gamma(x)$ we see

$(b) = (a)$ and $(a) - (b) = 0$!

(b) ⓐ $\sim \Gamma(1-\frac{d}{2})$ so ⓐ is convergent for $d<2$.

ⓑ $\sim \frac{-1}{1-\frac{d}{2}}\Gamma(2-\frac{d}{2})$ so ⓑ is convergent for $2<d<4$

In the limit $d \to 2-2\varepsilon$

ⓐ $\sim \Gamma(1-\frac{d}{2}) = \Gamma(\varepsilon) \sim \frac{1}{\varepsilon_{UV}}$

ⓑ $\sim \frac{-1}{1-\frac{d}{2}}\Gamma(2-\frac{d}{2}) \sim -\frac{1}{\varepsilon_{IR}}$

$$\therefore \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+i\varepsilon} \sim \frac{1}{\varepsilon}-\frac{1}{\varepsilon}=0\,!$$

<u>Problem 2</u>

$$S=\int d^4x\left[\bar{\psi}_0(i\not{\partial}-m_0-ie_0\not{A}_0)\psi_0-\frac{1}{4}(F_0^{\mu\nu})^2\right]$$

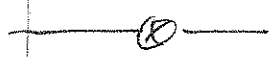
$d=4-2\varepsilon$

$\psi_0=\mu^{-\varepsilon}\sqrt{Z_2}\,\psi$, $A_0^\mu=\mu^{-\varepsilon}\sqrt{Z_3}\,A^\mu$

$m_0=Z_m m$, $e_0=\mu^{\varepsilon}Z_g\cdot e$

$$\Rightarrow S=\int d^4x\left[\bar{\psi}(i\not{\partial}-m-ie\not{A})\psi-\frac{1}{4}(F^{\mu\nu})^2\right]$$
$$+(Z_2-1)\bar{\psi}i\not{\partial}\psi-(Z_2Z_m-1)m\bar{\psi}\psi$$
$$-(Z_2\sqrt{Z_3}Z_g-1)e\bar{\psi}\not{A}\psi-(Z_3-1)\frac{1}{4}(F^{\mu\nu})^2\Big]$$

So we have the following counter terms



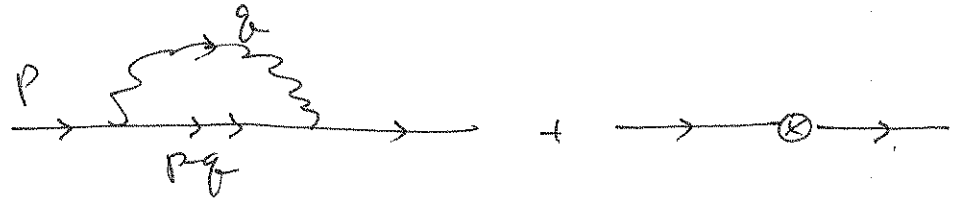
$i(\delta_2 \not{p} - \delta_m m)$. $\qquad \delta_2 = Z_2 - 1$, $\quad \delta_m = Z_2 Z_m - 1$

$-i\delta_3 (g^{\mu\nu} q^2 - q^\mu q^\nu)$ $\qquad \delta_3 = Z_3 - 1$

$-i\delta_1 e \gamma^\mu$. $\qquad \delta_1 = Z_2 \sqrt{Z_3}\, Z_g$

To determine δ_2 and δ_m, we compute the electron self-energy $-i\Sigma(\not{p})$ to $O(\alpha)$

$-i\Sigma_2(\not{p}) =$



(a) $\qquad$ (b).

$$(a) = -e^2 \mu^{2\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{\gamma^\mu (\not{p} - \not{q} + m)\gamma_\mu}{[(p-q)^2 - m^2 + i\varepsilon][q^2 + i\varepsilon]}$$

$$= -e^2 \int_0^1 dx\, \mu^{2\varepsilon} \int \frac{d^d \ell}{(2\pi)^d} \frac{(-2)((1-x)(1-\varepsilon)\not{p} + (4-2\varepsilon) m}{(\ell^2 - \Delta + i\varepsilon)^2}$$

$$\left\{ \begin{array}{l} \Delta = x m^2 - x(1-x) p^2 \\ \ell = q - x p \end{array} \right\}$$

$$= \frac{-ie^2}{(4\pi)^2} \Gamma(\varepsilon) \int_0^1 dx \left(\frac{4\pi\mu^2}{\Delta}\right)^\varepsilon \left[-2(1-\varepsilon)(1-x)\not{p} + (4-2\varepsilon) m\right]$$

$$(b) = \text{c.t.} = i(\delta_2 \not{p} - \delta_m m)$$

$$\Rightarrow \Sigma_2(\not p) = \frac{\alpha}{4\pi}\frac{1}{\varepsilon}(1-\varepsilon r_E+\cdots)\int_0^1 dx\left[1+\varepsilon\log\frac{4\pi\mu^2}{\Delta}\right]\left[-2(1-\varepsilon)(1-x)\not p + (4-2\varepsilon)m\right]$$

$$+(-\delta_2\not p+\delta_m\cdot m)$$

$$=\frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon}-r_E+\log 4\pi\right)\int_0^1 dx\,\log\frac{\mu^2}{\Delta}\left[-2(1-x)\not p+4m\right]$$

$$+\frac{\alpha}{4\pi}\int_0^1 dx\left[2(1-x)\not p-2m\right]$$

$$+(-\delta_2\not p+\delta_m m).$$

$$=\frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon}-r_E+\log 4\pi\right)\left[-\not p+4m\right]$$

$$+\frac{\alpha}{4\pi}\int_0^1 dx\,\log\frac{xm^2-x(1-x)p^2}{\mu^2}\left[2(1-x)\not p-4m\right]$$

$$+\frac{\alpha}{4\pi}\left[\not p-2m\right]$$

$$-\delta_2\not p+\delta_m m.$$

In $\overline{MS}$ scheme we require the c.t. to remove $\left(\frac{1}{\varepsilon}-r_E+\log 4\pi\right)$ part of the divergence.

$$\Rightarrow \delta_2 = Z_2-1 = -\frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon}-\cancel{4\pi}\, r_E+\log 4\pi\right)$$

$$\delta_m = Z_2Z_m-1 = -\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon}-r_E+\log 4\pi\right).$$

$$\Rightarrow Z_2 = 1-\frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon}-r_E+\log 4\pi\right)$$

$$Z_m = 1-\frac{3\alpha}{4\pi}\left(\frac{1}{\varepsilon}-r_E+\log 4\pi\right)$$

To compute δ_3 we need the photon self-energy

$$i\Pi^{\mu\nu}(q^2) = -e^2\mu^{2\varepsilon}\int \frac{d^dk}{(2\pi)^d}\,\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{q}+m)\gamma^\nu(\not{k}+m)\right]}{\left[(k-q)^2-m^2+i\varepsilon\right]\left[k^2-m^2+i\varepsilon\right]}$$

$$-i\delta_3\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)$$

$$\Pi^{\mu\nu} = \frac{2\alpha}{\pi}\Gamma(\varepsilon)\left(g^{\mu\nu}q^2-q^\mu q^\nu\right)\int_0^1 dx\, x(1-x)\left[\frac{m^2-x(1-x)q^2}{4\pi\mu^2}\right]^\varepsilon$$

$$+\frac{\alpha}{\pi}\left(g^{\mu\nu}q^2-q^\mu q^\nu\right)\left(-\frac{1}{3\varepsilon}+\log 4\pi - \gamma_E\right).$$

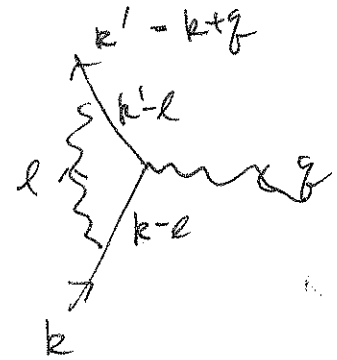
$$-\ \delta_3\left(g^{\mu\nu}q^2-q^\mu q^\nu\right).$$

$$\Rightarrow\ \delta_3 = Z_3 - 1 = -\frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon}-\gamma_E+\log 4\pi\right)$$

or $$Z_3 = 1-\frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon}-\gamma_E+\log 4\pi\right).$$

To compute δ_1, we need Γ^μ

$e\Gamma^\mu =$



$+$

$-i\delta_1 e\gamma^\mu$

$$\Gamma^\mu = -ie^2\mu^{2\varepsilon}\int\frac{d^dl}{(2\pi)^d}\frac{\gamma^\alpha(\not k'-\not l+m)\gamma^\mu(\not k-\not l+m)\gamma_\alpha}{[(k'-l)^2-m^2][(k-l)^2-m^2][l^2+i\varepsilon]} - i\delta_1\gamma^\mu$$

$$= -2ie^2\int_0^1 dx\int_0^{1-x}dy\,\mu^{2\varepsilon}\int\frac{d^dl}{(2\pi)^d}\frac{\gamma^\alpha(\not k'-\not l+m)\gamma^\mu(\not k-\not l+m)\gamma_\alpha}{(\tilde l^2-\Delta)^3}$$

$$-i\delta_1\gamma^\mu.$$

$$\tilde l = l - xk' - yk; \quad \Delta = -x(1-x-y)k'^2 - y(1-x-y)k^2 - xyq^2 + (x+y)m^2.$$

(Skipping the algebras ...)

$$= (-i)\frac{\alpha}{2\pi}\gamma^\mu\int_0^1 dx\int_0^{1-x}dy\,\Gamma(\varepsilon)\cdot\left[\frac{\Delta}{4\pi\mu^2}\right]^{-\varepsilon}$$

$$-i\delta_1\gamma^\mu \quad + (\text{terms finite as } \varepsilon\to 0)$$

$$= (-i)\frac{\alpha}{4\pi}\gamma^\mu\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi\right) + \cdots$$

$$-i\delta_1\gamma^\mu.$$

$$\therefore\ \delta_1 = Z_2\sqrt{Z_3}Z_g - 1 = -\frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi\right)$$

$$\Rightarrow \quad Z_g = 1 + \frac{1}{6}\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon} - r_E + \log 4\pi\right)$$

To summarize :

$$Z_2 = 1 - \frac{1}{4}\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon} - r_E + \log 4\pi\right)$$

$$Z_3 = 1 - \frac{1}{3}\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon} - r_E + \log 4\pi\right)$$

$$Z_m = 1 - \frac{3}{4}\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon} - r_E + \log 4\pi\right)$$

$$Z_g = 1 + \frac{1}{6}\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon} - r_E + \log 4\pi\right).$$

We see that, indeed,

$$Z_1 = Z_2\sqrt{Z_3}\, Z_g = Z_2$$

or $\quad Z_g = \dfrac{1}{\sqrt{Z_3}}.$

The ward identity is satisfied.

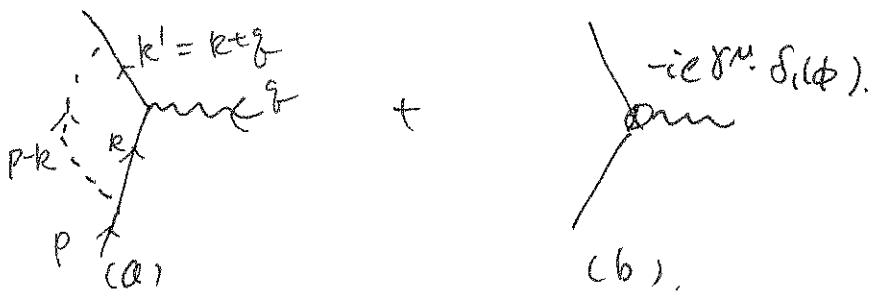
Problem 3. (7.3 of Peskin & Schroeder)

$$H_{int} = \int d^3x \frac{\lambda}{\sqrt{2}} \phi \bar{\psi}\psi + \int d^3x\, e\bar{\psi}\not{A}\psi$$

I will use MS scheme for renormalization!!

(a)

$e\delta\Gamma^\mu =$



$\longrightarrow$ to compensate for dim of λ.

$$(a) = \mu^{2\varepsilon}\left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{((p-k)^2-M^2+i\varepsilon)} \bar{u}(p') \frac{i(\not{k}'+m)}{(k'^2-m^2+i\varepsilon)} \gamma^\mu \frac{i(\not{k}+m)}{(k^2-m^2+i\varepsilon)} u(p)$$

$$= i\mu^{2\varepsilon}\lambda^2 \int_0^1 dx\,dy\,dz\, \delta(x+y+z-1) \int \frac{d^d\ell}{(2\pi)^d} \frac{N}{(\ell^2-\Delta+i\varepsilon)^3}$$

$$\Delta = -xy\,q^2 + (1-z)^2 m^2 + zM^2.$$

$$\ell = k + (yq - zp)$$

$$N = \bar{u}(p')\left[\not{\ell} + z\not{p} + (1-y)\not{q} + m\right]\gamma^\mu\left[\not{\ell} + z\not{p} - y\not{q} + m\right]u(p)$$

Since we are only interested in the UV-divergence, we can set $q=0$, and use Dirac eq. to simplify N:

$$N = \bar{u}(p')\not{\ell}\gamma^\mu\not{\ell}u(p) + \bar{u}(p')\left[z\not{p}+m\right]\gamma^\mu\left[z\not{p}+m\right]u(p)$$

$$\longrightarrow \bar{u}(p)\gamma^\mu u(p)\left[\ell^2\frac{2-d}{d} + (1+z)^2 m^2\right]$$

$$\therefore\ (a) = \mu^{2\varepsilon} i\lambda^2 \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2 \frac{2-d}{d} + ((1-z)^2 m^2}{(\ell^2-\Delta+i\varepsilon)^3} \cdot \bar{u}(p)\gamma^\mu u(p)$$

keeping only UV-div term

$$i\mu^{2\varepsilon}\lambda^2 \int dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2 \cdot \frac{2-d}{d}}{(\ell^2-\Delta+i\varepsilon)^3}\ \bar{u}(p)\gamma^\mu u(p)$$

$$= \frac{-\lambda^2}{2\cdot(4\pi)^2}\ \bar{u}(p)\gamma^\mu u(p) \int_0^1 dx \int_0^{1-x} dy \left(\frac{4\pi\mu^2}{\Delta}\right)^\varepsilon \Gamma(\varepsilon)\frac{(2+2\varepsilon)}{2}$$

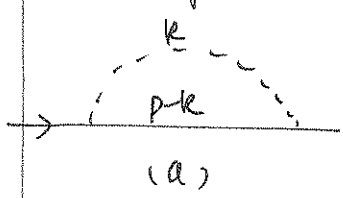
$$= \frac{+\lambda^2}{2\cdot(4\pi)^2}\ \bar{u}(p)\gamma^\mu u(p)\cdot\frac{1}{2}\Big(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \underbrace{\cdots\cdots}_{\text{finite as } \varepsilon\to 0}\Big).$$

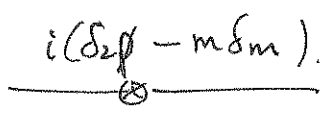
$$(b) = i\,\delta_1\,\bar{u}(p)\gamma^\mu u(p)$$

$$\Rightarrow\quad \delta_1 = -\frac{1}{4}\frac{\lambda^2}{16\pi^2}\frac{1}{\varepsilon} = Z_1 - 1.$$

or $\quad Z_1 = 1 - \frac{1}{4}\frac{\lambda^2}{16\pi^2}\frac{1}{\varepsilon}.$

Now compute Z_2:

$-i\Sigma_2 =$ 

(a) $\quad + \quad$ 

(b)

$$(a) = \left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{i}{[(p-k)^2 - M^2 + i\varepsilon]} \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}$$

$$= \frac{i\lambda^2}{2}\mu^{2\varepsilon}\int dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\not{\ell} + x\not{p} + m}{(\ell^2-\Delta)^2}$$

$$\ell = k - xp\ ;\quad \Delta = -x(1-x)p^2 + xM^2 + (1-x)m^2$$

To obtain δ_2 we only need to focus on the coefficient of $\not{p}$:

$$\frac{i\lambda^2}{2}\mu^{2\varepsilon}\int dx \int \frac{d^d\ell}{(2\pi)^d}\frac{x\not{p}}{(\ell^2-\Delta)^2}$$

finite as $\varepsilon \to 0$

$$= \frac{i\lambda^2}{2(4\pi)^2}\int_0^1 dx\cdot x\cdot \not{p}\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \cdots\right)$$

$$= \frac{i\lambda^2}{4\cdot(16\pi^2)}\frac{1}{\varepsilon}\not{p} + \cdots$$

$$(b) = i\delta_2\not{p}$$

$$\Rightarrow \quad \delta_2 = Z_2 - 1 = -\frac{1}{4}\frac{\lambda^2}{16\pi^2}\frac{1}{\varepsilon}.$$

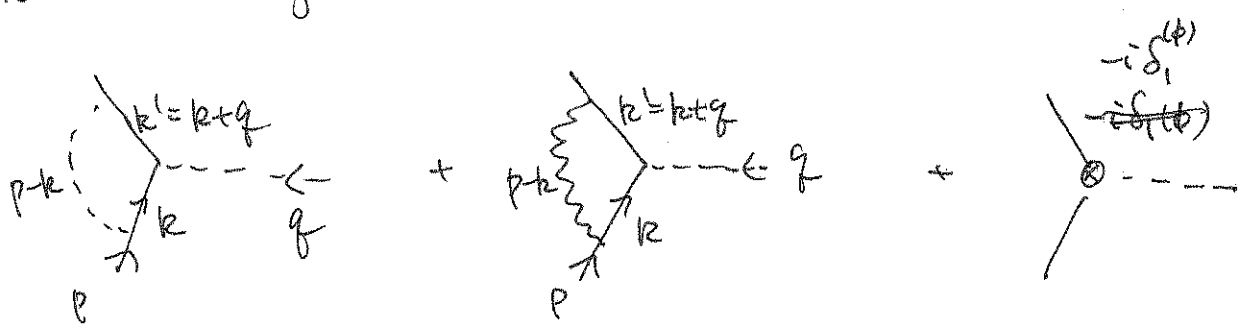
$$\text{or} \quad Z_2 = 1 - \frac{1}{4}\frac{\lambda^2}{16\pi^2}\frac{1}{\varepsilon}.$$

We then see $Z_1 = Z_2$!!

The Ward identity is true at $O(e^2)$.

<u>Part (b)</u>

The two diagrams are



$$\delta\Gamma = \left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \bar{u}(p') \frac{i}{((p-k)^2 - M^2 + i\varepsilon)} \frac{i(\not{k}+\not{q}+m)}{((k+q)^2 - m^2 + i\varepsilon)} \frac{i(\not{k}+m)}{(k^2 - m^2 + i\varepsilon)} u(p)$$

$$+ (-ie)^2 \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \bar{u}(p') \frac{i(\not{k}+\not{q}+m)}{((k+q)^2 - m^2)} \frac{-i}{((p-k)^2 - \mu_\gamma^2)} \frac{i(\not{k}+m)}{(k^2 - m^2)} u(p)$$

↳ tiny photon mass

Again set $q=0$ since we are only interested in the UV behavior.

Use $\ell = k - zp$, $\Delta_1 = (1-z)^2 m^2 + z M^2$, $\Delta_2 = (1-z)^2 m^2 + z \mu_\gamma^2$.

$$\delta\Gamma = \int_0^1 dx\,(1-x) \int \frac{d^d\ell}{(2\pi)^d} \bar{u}(p) \left[i\lambda^2 \frac{\ell^2 + (1+z)^2 m^2}{(\ell^2 - \Delta_1 + i\varepsilon)^3} - \alpha i e^2 \frac{d\ell^2 + m^2[d(z^2+1) + 2z(2-d)]}{(\ell^2 - \Delta_2 + i\varepsilon)^3} \right] \times u(p)$$

keeping only UV-div terms

$$\int_0^1 dx\,(1-x) \int \frac{d^d\ell}{(2\pi)^d} \bar{u}(p) \left[i\lambda^2 \frac{\ell^2}{(\ell^2 - \Delta_1 + i\varepsilon)^3} - \alpha i e^2 \frac{d \cdot \ell^2}{(\ell^2 - \Delta_2 + i\varepsilon)^3} \right] u(p)$$

$$= -\int_0^1 dx\,(1-x) \left(\frac{\lambda^2}{16\pi^2} - \frac{2\alpha}{\pi} \right) \cdot \frac{1}{\varepsilon} + (\text{finite as } \varepsilon \to 0)$$

$$\Rightarrow \quad \delta_1^{(\phi)} = + \frac{1}{2} \left(\frac{\lambda^2}{16\pi^2} - \frac{2\alpha}{\pi} \right) \frac{1}{\varepsilon}.$$

Compare w/ δ_2, computed in Problem 2 and 3(a) in this homework:

$$\delta_2 = -\frac{1}{4} \frac{\lambda^2}{16\pi^2} - \frac{1}{4} \frac{\alpha}{\pi}, \quad \text{we see } \delta_2 \neq \delta_1^{(\phi)} \text{ !!}$$

This is because $\phi\bar{\psi}\psi$ vertex is <u>NOT</u> a conserved current!