QFT II, Winter 2009

Solutions for HW #6

Problem 1

The vertex $\bar{\psi}\gamma_\mu\psi \cdot A^\mu$ satisfies the Ward identity, which ensures $Z_1 = Z_2$.

Problem 2. (Peskin & Schroeder 10.2)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 + \bar{\psi}_0(i\not{\partial} - M_0)\psi_0 - ig_0\bar{\psi}_0\gamma^5\psi_0\phi_0$$

Parity: $\psi \to \gamma^0\psi$, $\phi \to -\phi$.

(a) The superficial degree of divergence

$$D = 4 - \frac{3}{2}N_F - N_B$$

N_F = # of external fermions
N_B = # of ext. bosons

The following amplitudes are superficially divergent:

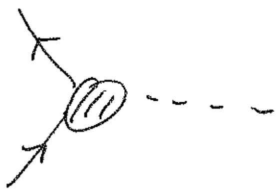
① $N_F = N_B = 0$, $D = 4$: vacuum bubbles, which we ignore.

② $N_F = 0$, $N_B = 1$, $D = 3$ and $N_F = 0$, $N_B = 3$, $D = 0$: these are boson 1-pt and 3-pt amplitudes. They must be zero because they are <u>NOT</u> invariant under parity $\phi \to -\phi$.

3 $N_F = 0,\ N_B = 2,\ D = 2$

4 $N_F = 2,\ N_B = 0,\ D = 1$

5 $N_F = 2,\ N_B = 1,\ D = 0.$



6 $N_F = 0,\ N_B = 4,\ D = 0$

Counter terms: 3 requires counter terms of the form

$$\delta_\phi \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \delta_m \frac{1}{2}\phi^2 m^2$$

This is already present in $\mathcal{L}$.

4 requires $\delta_2 i\bar{\psi}\not{\partial}\psi - \delta_M M\bar{\psi}\psi$ $\rightarrow$ already in $\mathcal{L}$

5 requires $-i\delta_g g\bar{\psi}\gamma^5\psi\phi$ $\rightarrow$ already in $\mathcal{L}$

6 requires $-\frac{\delta_\lambda}{4!}\phi^4$ $\rightarrow$ NOT present in $\mathcal{L}$!!

Need to include ~~$\mathcal{L}_5$~~ $\delta\mathcal{L} = \frac{\lambda_0}{4!}\phi_0^4$

Define $\phi_0 = Z_\phi \phi$, $m_0^2 = Z_m m^2$,

$\psi_0 = Z_2 \psi$, $M_0 = Z_M M$, $g_0 = Z_g g$, $\lambda_0 = Z_\lambda \lambda$

$$\mathcal{L} + \delta\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) + \bar\psi(i\partial\!\!\!/ - M)\psi - ig\bar\psi\gamma_5\psi\phi - \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{2}(\delta_\phi \partial_\mu\phi\partial^\mu\phi - \delta_m m^2 \phi^2) + \bar\psi(\delta_2 i\partial\!\!\!/ - \delta_M M)\psi - i\delta_g g\bar\psi\gamma_5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4$$

$\delta_\phi = Z_\phi - 1$, $\delta_m = Z_\phi Z_m - 1$, $\delta_2 = Z_2 - 1$, $\delta_M = Z_2 Z_M - 1$.

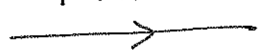
$\delta_g = Z_2\sqrt{Z_\phi}\cdot Z_g - 1$. $\delta_\lambda = Z_\phi^2 Z_\lambda - 1$.

Feynman rules :

$\frac{i}{p^2 - m^2 + i\varepsilon}$

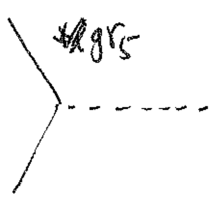
$\cancel{i(\delta_\phi p}$ $i(\delta_\phi p^2 - \delta_m m^2)$.

$\frac{i}{p\!\!\!/ - M + i\varepsilon}$

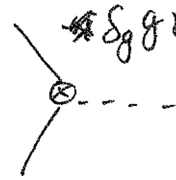


$i(p\!\!\!/\delta_2 - M\delta_M)$

$g\gamma_5$



$\delta_g g \gamma_5$

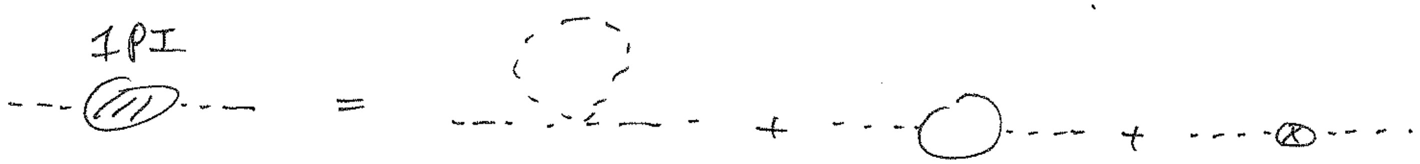


$-i\lambda$

$-i\delta_\lambda\lambda$.

(b) We will compute the counter terms in MS scheme. also $d = 4 - 2\varepsilon$

look at <u>Scalar 2-pt function first</u>



$$-i\mathcal{M} = \frac{-i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\varepsilon} - g^2\int\frac{d^dk}{(2\pi)^d}\frac{Tr[(\not k+M)\gamma_5(\not k+\not p+M)\gamma_5]}{[(k+p)^2-M^2][k^2-M^2]}$$

symmetry factor

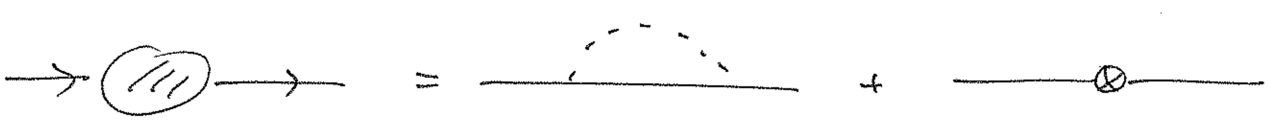
$$+\,i(p^2\delta_\phi - m^2\delta_m)$$

→ a closed fermion loop gets a "-1". See (4.120) in P&S.

The computation is straightforward.

$$\delta_\phi = -\frac{g^2}{4\pi^2}\frac{1}{(2\varepsilon)}, \quad \delta_m = \left(\frac{\lambda}{16\pi^2} - \frac{g^2}{2\pi^2}\frac{M^2}{m^2}\right)\frac{1}{(2\varepsilon)}$$

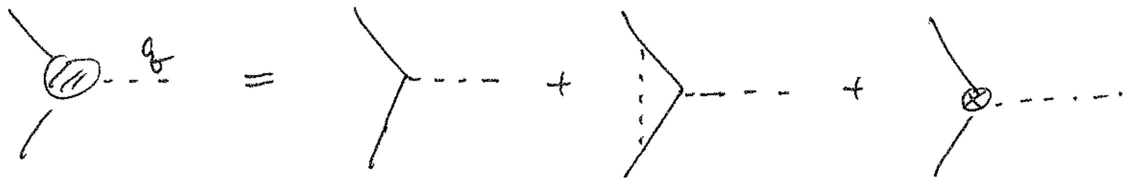
<u>Fermion 2-pt function</u>



$$-i\Sigma(\not p) = g^2\int\frac{d^dk}{(2\pi)^d}\frac{i}{[(p-k)^2-m^2]}\frac{\gamma_5\, i(\not k+M)\gamma_5}{[k^2-M^2]} + i(\not p\,\delta_2 - M\delta_M)$$

$$\Rightarrow \delta_2 = -\frac{g^2}{16\pi^2}\frac{1}{(2\varepsilon)}, \quad \delta_M = \frac{-g^2}{8\pi^2}\frac{1}{(2\varepsilon)}$$

3-pt function



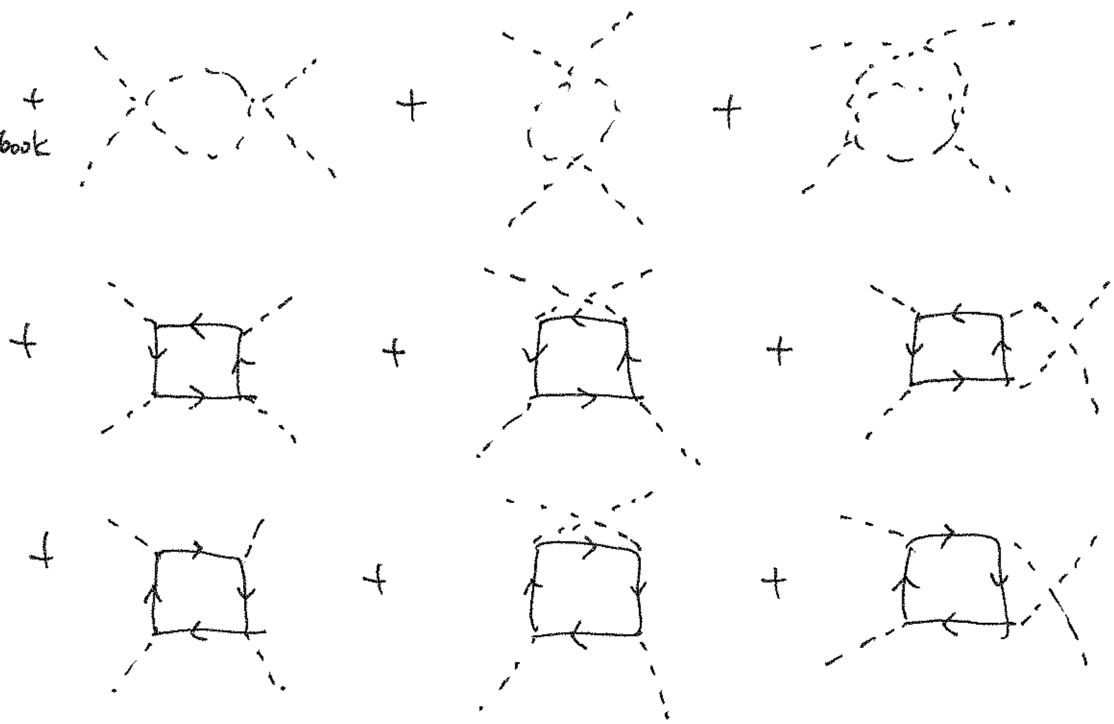
We can set $q=0$ if we're only interested in UV divergences.

$$\delta\Gamma^5(q=0) = -ig^2\int\frac{d^dk}{(2\pi)^d}\frac{\gamma_5(\not{k}+M)\gamma_5(\not{k}+M)\gamma_5}{[(p-k)^2-m^2][k^2-M^2][k^2-M^2]} + \delta_g g\,\gamma_5$$

$$\Rightarrow \quad \delta_g\cdot g = +\frac{g^2}{8\pi^2}\frac{1}{(2\varepsilon)}.$$

4-pt function

These are computed in textbook



The arrow in the fermion line is different!

$$-i\mathcal{M} = \underbrace{\frac{i\,3\lambda^2}{16\pi^2}\frac{1}{(2\varepsilon)}}_{(10.23)\text{ in PRS}} - i\delta_\lambda \cdot \lambda$$

$$-6g^4 \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^5(\not{k}+M)\gamma^5(\not{k}-\not{p}_1+M)\gamma_5(\not{k}-\not{p}_1-\not{p}_2+M)\gamma_5(\not{k}-\not{p}_1-\not{p}_2-\not{p}_3+M)\right]}{(k^2-M^2)((k-p_1)^2-M^2)((k-p_1-p_2)^2-M^2)((k-p_1-p_2-p_3)^2-M^2)}$$

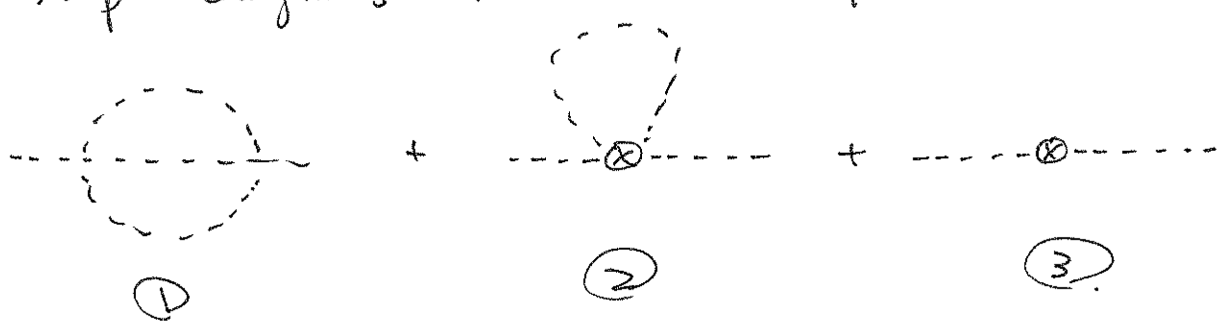
Symmetry factor

$$= \frac{i\,3\lambda^2}{16\pi^2}\frac{1}{(2\varepsilon)} - i\frac{3g^4}{\pi^2}\frac{1}{(2\varepsilon)} - i\delta_\lambda\cdot\lambda$$

$$\therefore\ \delta_\lambda\cdot\lambda = \left(\frac{3\lambda^2}{16\pi^2} - \frac{3g^4}{\pi^2}\right)\frac{1}{(2\varepsilon)}$$

Problem 3 (10.3 in Peskin & Schroeder)

Two-loop diagrams in massless ϕ^4



$$① = \frac{1}{3!}\mu^{4\varepsilon}(-i\lambda)^2 \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{i}{k_1^2}\frac{i}{k_2^2}\frac{i}{(p-k_1-k_2)^2}.$$

symmetry factor

$$= \frac{1}{3!}\mu^{4\varepsilon} i\lambda^2 \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dl_1}{(2\pi)^d}\int_0^1 dx\, \frac{1}{k_1^2}\frac{1}{(l_1^2-\Delta_1+i\varepsilon)^2} \qquad l_1 = k_2 - x(p-k_1),\quad \Delta_1 = x(x-1)(p-k_1)^2$$

$$= \frac{1}{3!} i\lambda^2 \mu^{4\varepsilon}\frac{i}{(4\pi)^{\frac{d}{2}}}\Gamma(2-\tfrac{d}{2})\int_0^1 dx\,\frac{1}{[x(1-x)]^{2-\frac{d}{2}}}\int\frac{d^dl_2}{(2\pi)^d}\int_0^1 dy\,\frac{\Gamma(3-\frac{d}{2})}{\Gamma(2-\frac{d}{2})}\frac{y^{1-\frac{d}{2}}}{(l_2^2-\Delta_2+i\varepsilon)^{3-\frac{d}{2}}}$$

$$l_2 = k_1 - yp,\quad \Delta_2 = yp^2(y-1)$$

$$= \frac{1}{3!} i\lambda^2 \mu^{4\varepsilon} \left[\frac{+6}{(4\pi)^{\frac{d}{2}}} \right]^2 \frac{\Gamma(3-d)}{(p^2)^{3-d}} \int_0^1 dx \frac{1}{[x(1-x)]^{2-\frac{d}{2}}}$$

$$\times \int_0^1 dy \frac{y^{1-\frac{d}{2}}}{[y(y-1)]^{3-d}}$$

$$\rightarrow -ip^2 \frac{\lambda^2}{\cancel{24}\,12 \cdot (4\pi)^4} \left[\left(\frac{-1}{2\varepsilon} \right) + \log p^2 + \cdots \right].$$

$\hookrightarrow$ we used $d = 4 - 2\varepsilon$ instead of $d = 4 - \varepsilon$ here.

$$(2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = 0 \quad \text{in DR !!}$$

$$(3) = i\delta_\phi^{(2)} p^2$$

$\hookrightarrow$ δ_ϕ at 2-loop order.

$$\therefore \quad \delta_\phi^{(2)} = \frac{\lambda}{12(4\pi)^4} \frac{-1}{(2\varepsilon)}$$

Problem 4

$$\phi_0 = \mu^{-2\varepsilon} \sqrt{Z_\phi}\, \phi_r(\mu) = (\mu')^{-2\varepsilon} \sqrt{Z_\phi}\, \phi_r(\mu')$$

write $\phi_r(\mu') = \xi\, \phi_r(\mu)$. where ξ is finite.

then $G_N(\mu') \equiv \langle \Omega | T \phi_1(\mu') \cdots \phi_N(\mu') | \Omega \rangle$

$$= \xi^N \langle \Omega | T \phi_1(\mu) \cdots \phi_N(\mu) | \Omega \rangle$$

$$= \xi^N G_N(\mu).$$

By LSZ reduction an S-matrix element is given by

$$S = \lim_{p^2 \to m_{ph}^2} \frac{G_N \cdot \hat{Z}^{\frac{N}{2}}}{\prod_{i=1}^{N} G_2}$$

$\hat{Z} \to$ residue at $p^2 = m_{ph}^2$.

$\to$ dividing out the external legs

But since $G_2(\mu) = \xi^{-2} G_2(\mu')$

and the location of the pole m_{ph}^2 is independent of RG scale $\mu \Rightarrow \hat{Z}' = \xi^2 \hat{Z}$

$$\Rightarrow S = \lim_{p^2 \to m_{ph}^2} \frac{(\xi^{-N} G_N')(\xi^{-2} \hat{Z}')^{\frac{N}{2}}}{\prod_{i=1}^{N} (\xi^{-2} G_2')}$$

$$= \lim_{p^2 \to m_{ph}^2} \frac{G_N' \cdot (\hat{Z}')^{\frac{N}{2}}}{\prod_{i=1}^{N} G_2'}$$

i.e., S is invariant as claimed as $\mu \to \mu'$.