QFT II Winter 2009

HW#7 Solutions

Problem 1 (P&S 12.1)

Sol'n: The pseudo-scalar Yukawa theory has been studied in Prob2 (P&S 10.2) in HW#6. There we found

$$Z_\phi = 1+\delta_\phi = 1-\frac{g^2}{8\pi^2}\frac{1}{\varepsilon}\ ;\ Z_2 = 1+\delta_2 = 1-\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}$$

$$Z_g = \frac{1+\delta_g}{\sqrt{Z_\phi}\,Z_2} = \left(1-\frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\right)\left(1+\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}\right)\left(1+\frac{1}{2}\frac{g^2}{8\pi}\frac{1}{\varepsilon}\right)$$

$$= 1+\frac{g^2}{32\pi^2}\frac{1}{\varepsilon}.$$

$$Z_\lambda = \frac{1+\delta_\lambda}{Z_\phi^2} = \left[1+\frac{1}{\lambda}\left(\frac{3\lambda^2}{32\pi^2}-\frac{3g^4}{2\pi^2}\right)\frac{1}{\varepsilon}\right]\left[1+\frac{g^2}{4\pi^2}\frac{1}{\varepsilon}\right]$$

$$= \left[1+\left(\frac{3\lambda}{32\pi^2}-\frac{3g^4}{2\pi^2\lambda}+\frac{g^2}{4\pi^2}\right)\frac{1}{\varepsilon}\right]$$

We see that $Z_g = Z_g(\varepsilon, g)$.

Since $g_0 = \mu^\varepsilon Z_g g$, $\mu\frac{d}{d\mu}g_0 = 0 \Rightarrow \varepsilon g Z_g + \mu\frac{d}{d\mu}(Z_g g) = 0$.

$$\Rightarrow \varepsilon g Z_g + \mu\frac{d}{d\mu}g\cdot\frac{\partial}{\partial g}(gZ_g) = 0.$$

Let's write $Z_g = 1+\sum_{n=1}^{\infty}\frac{Z_g^{(n)}}{\varepsilon^n}$ and $gZ_g = g+\sum_{n=1}^{\infty}\frac{a_g^{(n)}}{\varepsilon^n}$.

$$\Rightarrow \varepsilon\left(g+\sum_{n=1}^{\infty}\frac{a_g^{(n)}}{\varepsilon^n}\right)+\beta(g,\varepsilon)\left[1+\sum_{n=1}^{\infty}\frac{1}{\varepsilon^n}\frac{\partial}{\partial g}a_n\right]=0$$

$$\Rightarrow \beta(g,\varepsilon) = -\varepsilon g - a_1 + g\frac{\partial}{\partial g}a_1 = -\varepsilon g + g^2\frac{\partial}{\partial g}Z_g^{(1)}$$

$$\therefore\ \beta(g,0) = \lim_{\varepsilon\to 0}\beta(g,\varepsilon) = g^2\frac{\partial}{\partial g}z_g^{(1)} = \frac{g^3}{16\pi^2}$$

On the other hand, $\lambda_0 = \mu^{2\varepsilon} Z_\lambda \cdot \lambda$, and $Z_\lambda = Z_\lambda(\varepsilon, g, \lambda)$

$$\Rightarrow\ 2\varepsilon\cdot\lambda Z_\lambda + \mu\frac{d}{d\mu}(\lambda Z_\lambda)$$

$$= 2\varepsilon\lambda Z_\lambda + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}(\lambda Z_\lambda) + \mu\frac{\partial}{\partial\mu}\lambda\frac{\partial}{\partial\lambda}(\lambda Z_\lambda) = 0$$

$$\Rightarrow 2\varepsilon\left(\lambda + \sum_{n=1}^{\infty}\frac{a_\lambda^{(n)}}{\varepsilon^n}\right) + \beta(g,\varepsilon)\frac{\partial}{\partial g}\left(\lambda + \sum_{n=1}^{\infty}\frac{a_\lambda^{(n)}}{\varepsilon^n}\right)$$

$$+\ \beta(\lambda,\varepsilon)\frac{\partial}{\partial\lambda}\left(\lambda + \sum_{n=1}^{\infty}\frac{a_\lambda^{(n)}}{\varepsilon^n}\right) = 0$$

$$\therefore\ \beta(\lambda,\varepsilon) \cong \frac{-2\varepsilon\left(\lambda + \frac{a_\lambda^{(1)}}{\varepsilon}\right) - (-\varepsilon g)\frac{1}{\varepsilon}a_\lambda^{(1)}}{1 + \frac{1}{\varepsilon}\frac{\partial}{\partial\lambda}a_\lambda^{(1)}}$$

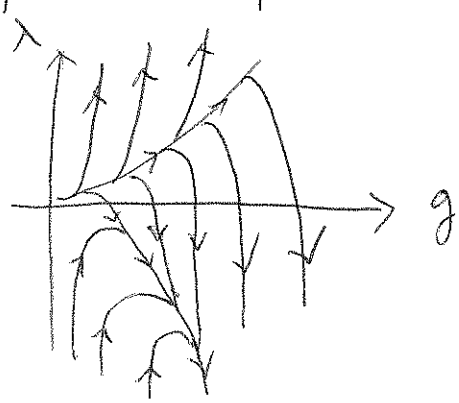
$$= -2\varepsilon\lambda + 2\lambda^2\frac{\partial}{\partial\lambda}z_\lambda^{(1)} + g\lambda\frac{\partial}{\partial g}z_\lambda^{(1)}$$

$$z_\lambda^{(1)} = \frac{3\lambda}{32\pi^2} - \frac{3g^4}{2\pi^2\lambda} + \frac{g^2}{4\pi^2} \Rightarrow \beta(\lambda,0) = \frac{3\lambda^2 - 48g^4 + 8g^2\lambda}{16\pi^2}$$

$$\therefore\ \beta_g(g,\lambda) = \frac{g^3}{16\pi^2}$$

$$\beta_\lambda(g,\lambda) = \frac{3\lambda^2 - 48g^4 + 8g^2\lambda}{16\pi^2}$$

One can use a software to plot the vector field (β_g, β_λ):



Problem 2

(a) For $Z_{g,m} = Z^{(0)}_{g,m}(\varepsilon, e) + \sum_{n=1}^{\infty} \frac{Z^{(n)}_{g,m}(e) + C^{(n)}_{g,m}}{\varepsilon^n}$

For $\beta(e,\varepsilon)$:

$$\mu \frac{d}{d\mu} e_0 = \mu \frac{d}{d\mu}(\mu^{\varepsilon} Z_g e) = 0$$

$$\Rightarrow \beta(e,\varepsilon) = \frac{-\varepsilon \cdot e Z_g}{\frac{\partial}{\partial e}(Z_g e)} \cong \frac{-\varepsilon\left(e Z_g^{(0)} + \frac{e Z_g^{(1)} + e C_g^{(1)}}{\varepsilon}\right)}{\frac{\partial}{\partial e}(e Z_g^{(0)}) + \frac{\frac{\partial}{\partial e}(e Z_g^{(1)}) + C_g^{(1)}}{\varepsilon}} + O(\varepsilon^2)$$

$$= \frac{1}{\frac{\partial}{\partial e}(e Z_g^{(0)})} \left\{ -\varepsilon e Z_g^{(0)} - (e Z_g^{(1)} + e C_g^{(1)}) + e Z_g^{(0)} \cdot \frac{\frac{\partial}{\partial e}(e Z_g^{(1)}) + C_g^{(1)}}{\frac{\partial}{\partial e}(e Z_g^{(0)})} \right\}$$

$\therefore$ In the most general case one doesn't get the same coefficients. However, for different mass-independent schemes such as MS & $\overline{MS}$,

$Z_g^{(0)} = 1$ in MS

$Z_g^{(0)} = 1 + O(g^2)$ in $\overline{MS}$

$\therefore \quad e Z_g^{(0)} = e + O(e^3)$ can be dropped at this order we're working at.

then $\beta(e,\varepsilon) = -\varepsilon e - (e Z_g^{(1)} + e C_g^{(1)}) + e \frac{\partial}{\partial e}(e Z_g^{(1)}) + e C_g^{(1)}$

$$= -\varepsilon e + e^2 \frac{\partial}{\partial e} Z_g^{(1)}$$

then we do get the same coefficients!

Next for $\gamma_m = \frac{\partial \log m}{\partial \log \mu} = -\frac{\partial \log Z_m}{\partial e}\frac{\partial e}{\partial \log \mu}$

$$= -\beta(e,\varepsilon)\frac{\partial}{\partial e}\log Z_m.$$

again drop the $O(g^2)$ contribution in $Z_m^{(0)}$ and set

$Z_m^{(0)} = 1.$

Then $\gamma_m = -\left(-\varepsilon e + e^2 \frac{\partial}{\partial g} Z_g^{(1)}\right)\frac{\partial}{\partial e}\log\left(1 + \sum_{n=1}^{\infty}\frac{Z_m^{(n)} + C_m^{(n)}}{\varepsilon^n}\right)$

$$= e\frac{\partial}{\partial e} Z_m^{(1)}$$

$\therefore$ We also get the same coefficient!

(b) In this case the total derivative

$$\mu\frac{d}{d\mu} = \mu\frac{\partial e}{\partial \mu}\frac{\partial}{\partial e} + \mu\frac{\partial\left(\frac{m}{\mu}\right)}{\partial \mu}\frac{\partial}{\partial\left(\frac{m}{\mu}\right)}$$

$$= \beta(e,\varepsilon)\frac{\partial}{\partial e} + \left(\frac{\partial m}{\partial \mu} - \frac{m}{\mu}\right)\frac{\partial}{\partial\left(\frac{m}{\mu}\right)}.$$

$$\mu\frac{d}{d\mu}e_0 = 0 \Rightarrow \varepsilon\left(e Z_g^{(0)} + \frac{e Z_g^{(1)}}{\varepsilon}\right) + \beta(g,\varepsilon)\frac{\partial}{\partial e}\left[e Z_g^{(0)} + \frac{1}{\varepsilon} e Z_g^{(1)}\right]$$

$$+ \left(\frac{\partial m}{\partial \mu} - \frac{m}{\mu}\right)\frac{\partial}{\partial\left(\frac{m}{\mu}\right)}\left[e Z_g^{(0)} + \frac{1}{\varepsilon} e Z_g^{(1)}\right] = 0.$$

To the order we're working, we can set $Z_g^{(0)} \sim 1$ if it's multiplied by $\beta(g,\varepsilon)$, since $\beta \sim -\varepsilon e + \cdots$

Furthermore, $\frac{\partial m}{\partial \mu} \sim \gamma_m \cdot \frac{m}{\mu} \sim O(e^2)\cdot\frac{m}{\mu}$. So can drop this term as well.

In the end we have

$$\beta(g,\varepsilon) = \frac{-\varepsilon e - e Z_g^{(1)} + \frac{m}{\mu}\frac{\partial}{\partial(\frac{m}{\mu})}(e Z_g^{(0)})}{1 + \frac{1}{\varepsilon}\frac{\partial}{\partial e}(e Z_g^{(1)})}$$

$$= -\varepsilon e + e^2 \frac{\partial}{\partial e} Z_g^{(1)} + \frac{m}{\mu}\frac{\partial}{\partial(\frac{m}{\mu})}(e Z_g^{(0)}).$$

$$\gamma_m = \mu \frac{d \log m}{d\mu} = -\frac{\partial e}{\partial \log \mu}\frac{\partial \log Z_m}{\partial e} - \left(\frac{\partial m}{\partial \mu} - \frac{m}{\mu}\right)\frac{\partial}{\partial(\frac{m}{\mu})}\log Z_m.$$

$Z_m^{(0)} \sim 1$ here!

$$= -(-\varepsilon e)\frac{\partial}{\partial e}\log\left[Z_m^{(0)} + \sum_{n=1}^{\infty}\frac{Z_m^{(n)}}{\varepsilon^n}\right]$$

$$+ \frac{m}{\mu}\frac{\partial}{\partial(\frac{m}{\mu})}\log\left[Z_m^{(0)} + \sum_{n=1}^{\infty}\frac{Z_m^{(n)}}{\varepsilon^n}\right]$$

$$= e\frac{\partial}{\partial e} Z_m^{(1)} + \cancel{\frac{m}{\mu}\frac{\partial}{\partial(\frac{m}{\mu})} Z_m}$$

$$+ \frac{m}{\mu}\underbrace{\frac{1}{Z_m^{(0)}}\frac{\partial}{\partial(\frac{m}{\mu})} Z_m^{(0)}}_{\sim 1}$$

$$= e\frac{\partial}{\partial e} Z_m^{(1)} + \frac{m}{\mu}\frac{\partial}{\partial(\frac{m}{\mu})} Z_m^{(0)}.$$

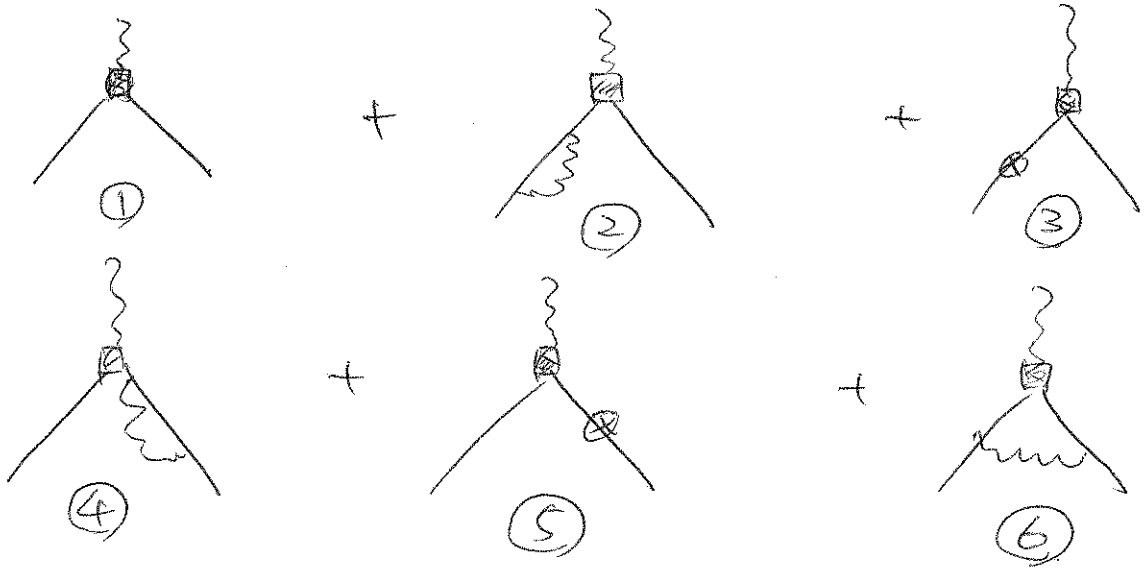
Problem 3

$$j^\mu = \bar{\psi}\gamma^\mu\psi(x) = Z_j^{-1}\bar{\psi}_0\gamma^\mu\psi_0 = Z_2 Z_j^{-1}\bar{\psi}\gamma^\mu\psi$$

(a)

$$\langle 0|T(\bar{\psi}(y)\psi(0)j^\mu(x))|0\rangle$$

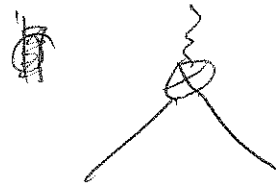
=



(b)

2 and 4 are UV-div, which are removed by the counter terms in 3 and 5

6 is also divergent, and needs a counter term



$= (Z_2 Z_j^{-1} - 1)$

But we already know the divergence in 6 is removed by $\delta_1 = Z_1 - 1 = Z_2 - 1$

$\Rightarrow Z_j = 1$.