QFT II, Winter 2009
HW#9 Solutions

Problem 1 (P&S 9.1)

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^*(D_\mu\phi) - m^2\phi^*\phi.$$

(a). We will go to the Euclidean space such that

$$\mathcal{L}_E = \frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^*(D^\mu\phi) + m^2\phi^*\phi \qquad \text{(contracted w/ } \delta_{\mu\nu}!\text{)}$$

$$= \mathcal{L}_E^{\text{free}} + \mathcal{L}_E^{\text{int.}}$$

$$\mathcal{L}_E^{\text{free}} = \frac{1}{4}(F_{\mu\nu})^2 + \partial_\mu\phi^*\partial^\mu\phi + m^2\phi^*\phi$$

$$\mathcal{L}_E^{\text{int}} = -ieA_\mu(\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi) + e^2A^2\phi^*\phi.$$

The complex scalar propagator is determined by

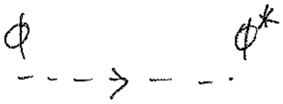
$$\int \mathcal{D}\phi^*\mathcal{D}\phi\, e^{-S_E^{\text{free}}[\phi,\phi^*] - J^*\phi - J\phi^*}$$

This is an integral w/ a quadratic form in the exponent, and can be performed using one of the identities derived in class:

$$\int dz^* dz\, e^{-(z^*Az) - b^*z - z^*b} = \frac{e^{b^*A^{-1}b}}{\det A}$$

A^{-1} in this case is given by $\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{1}{p_E^2+m^2}$

Which corresponds to the propagator



$$\frac{1}{p_E^2+m^2} \xrightarrow[\text{to Minkowski}]{\text{going back}} \frac{i}{p^2-m^2+i\varepsilon}$$

The photon propagator, on the other hand, has been derived in textbook

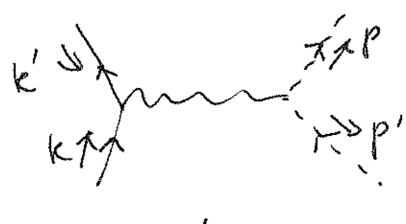
$$= \frac{-i}{p^2+i\varepsilon}\left(g_{\mu\nu}+(\xi-1)\frac{p_\mu p_\nu}{p^2}\right).$$

The interaction vertices can be read off as in before. (or see HW #10 in QFT I, Fall 2008!)

$-ie(p+q)_\mu$

$+2ie^2 g_{\mu\nu}$.

(b). There's only one diagram at the tree level for $e^+e^- \to \phi\phi^*$



Let's use the Feynman gauge $\xi = 1$:

$$iM = \left[\bar{v}(k')(+ie\gamma^\mu)u(k)\right] \times (-ie(p-p')^\nu) \times \frac{-ig_{\mu\nu}}{(k+k')^2} = \frac{e^2}{s}(p-p')^\mu \bar{v}\gamma_\mu u.$$

$$\frac{1}{4}\sum_s |M|^2 = \frac{e^4}{s^2}\left[2(k\cdot p - k\cdot p')(k'\cdot p - k'\cdot p') - (k\cdot k')(p-p')^2\right]$$

The kinematics are

$$S = (k+k')^2 = 4E_{cm}^2$$

$$k\cdot p = k'\cdot p' = E^2 - |\vec{k}||\vec{p}|\cos\theta \sim E^2(1-\beta\cos\theta)$$

$$k'\cdot p = k\cdot p' \simeq E^2(1+\beta\cos\theta).$$

$$k\cdot k' \approx 2E^2$$

$$(p-p')^2 = -4E_{cm}^2\beta^2. \qquad \beta = \frac{|\vec{p}|}{E} = \sqrt{1-\frac{m_\phi^2}{E^2}}$$

$$\Rightarrow \quad \frac{1}{4}\sum_s |M|^2 = \frac{1}{2}e^4\beta^2\sin^2\theta$$

$$\frac{d\sigma}{d\Omega_{cm}} = \frac{|\vec{p}|}{|\vec{k}|}\frac{1}{64\pi^2 S}\frac{1}{4}|M|^2 = \frac{\alpha^2}{8S}\beta^3\sin^2\theta$$

or $\quad \sigma = \dfrac{\pi\alpha^2}{3S}\times\beta^3$

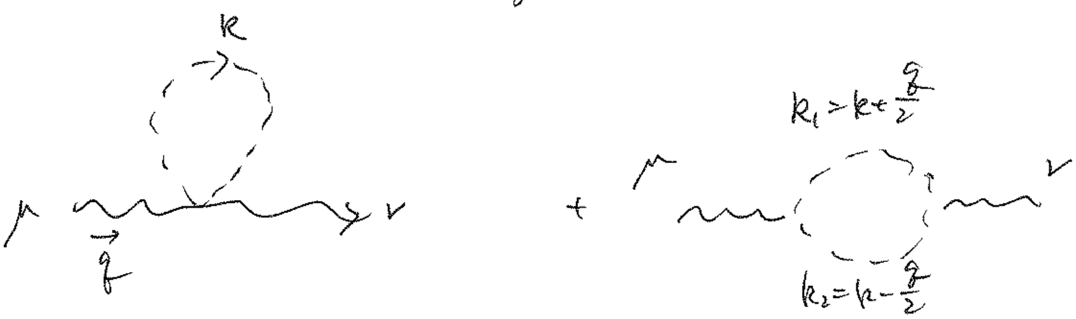
as for $e^+e^- \to \mu^+\mu^-$ (P&S 5.10, 5.13)

$$\frac{d\sigma}{d\Omega_{cm}} = \frac{\alpha^2}{4E^2}\beta\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\beta^2\cos^2\theta\right]$$

$$\sigma_{tot} = \frac{4\pi\alpha^2}{3E_{cm}^2}\beta\cdot\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right).$$

So Scalars prefer $\theta = 90^\circ$ while for $\mu^+\mu^-$ (or $q\bar{q}$) they prefer $\theta = 0$. Also near threshold $E \approx 2m_\phi, 2m_\mu$, the Scalar cross-section grows much slower due to β^3 suppression!

(c) There are two diagrams



$$iM^{\mu\nu} = \int \frac{d^dk}{(2\pi)^d} (2ie^2 g_{\mu\nu}) \frac{i}{k^2 - m_\phi^2} + \int \frac{d^dk}{(2\pi)^d} [-ie \cdot 2k_\mu][ie \cdot 2k_\nu] \frac{i}{k^2 - m_\phi^2} \frac{i}{k^2 - m_\phi^2}.$$

Following the hint from the textbook :

$$\int \frac{d^dk}{(2\pi)^d} (2ie^2 g_{\mu\nu}) \frac{i}{k^2 - m_\phi^2} = \int \frac{d^dk}{(2\pi)^d} \frac{(ie^2 g_{\mu\nu})\left\{ i\left[(k-\frac{q}{2})^2 - m_\phi^2\right] + i\left[(k+\frac{q}{2})^2 - m_\phi^2\right] \right\}}{\left[(k-\frac{q}{2})^2 - m_\phi^2\right]\left[(k+\frac{q}{2})^2 - m_\phi^2\right]}$$

$$\Rightarrow iM^{\mu\nu} = \int \frac{d^dk}{(2\pi)^d} \frac{-2e^2 g_{\mu\nu}\left(k^2 + \frac{1}{4}q^2 - m_\phi^2\right) + 4e^2 k_\mu k_\nu}{\left[(k-\frac{q}{2})^2 - m_\phi^2\right]\left[(k+\frac{q}{2})^2 - m_\phi^2\right]}$$

$$\frac{1}{\left[(k-\frac{q}{2})^2 - m_\phi^2\right]\left[(k+\frac{q}{2})^2 - m_\phi^2\right]} = \int_0^1 dx \frac{1}{\left[l^2 - \Delta + i\varepsilon\right]^2}.$$

Where $l = k + (x - \frac{1}{2})q$, $\Delta = -x(x-1)q^2 + m_\phi^2$.
$+ 2e^2 g_{\mu\nu} q^2 (x - \frac{1}{2})^2$

$$\Rightarrow iM^{\mu\nu} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-4e^2\left[g_{\mu\nu} q^2 - q^\mu q^\nu\right](x-\frac{1}{2})^2 - 2e^2 g_{\mu\nu}\left[l^2 + \frac{1}{4}q^2 - m_\phi^2\right] + 4e^2 \frac{l^2}{d} g_{\mu\nu}}{\left[l^2 - \Delta + i\varepsilon\right]^2}.$$

The terms that are not gauge invariant evaluates to zero.

and $iM^{\mu\nu} = i(g^{\mu\nu} q^2 - q^\mu q^\nu)\,\Pi(q^2)$

$$i\Pi(q^2) = -4e^2\int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{(x-\frac{1}{2})^2}{[l^2-\Delta]^2}$$

$$= -4e^2\int_0^1 dx\,(x-\tfrac{1}{2})^2 \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$\cancel{= -4i\int_0^1 dx\,(x-\tfrac{1}{2})^2}$$

$i\Pi(q^2) - i\Pi(q^2=0)$ is finite and the observed on-shell quantity

$$i\hat{\Pi} \equi i(\Pi(q^2)-\Pi(q^2=0)) = -\frac{i\alpha}{\pi}\int_0^1 dx\,(x-\tfrac{1}{2})^2 \log\left[\frac{m^2}{-x(1-x)q^2+m^2}\right]$$

In the limit $-q^2 \gg m^2$

$$i\hat{\Pi}(q^2) \longrightarrow -\frac{\alpha}{12\pi}\log\frac{m^2}{-q^2}.$$

which is $1/4$ of eq. (7.91) in P&S

(Fermion) $\hat{\Pi}_2(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\log\frac{m^2}{-x(1-x)q^2+m^2}$

$$\rightarrow -\frac{\alpha}{3\pi}\log\frac{m^2}{-q^2}.$$

Problem 2

$$Z[J] = \frac{1}{Z_0} e^{i\int d^4x\, V(-i\frac{\delta}{\delta J})} Z_{free}[J]$$

$$Z_{free}[J] = Z_0\, e^{-\frac{1}{2}\int d^4x d^4y\, J(x) D_F(x-y) J(y)}$$

$$V(\phi) = \lambda\phi^4/4!$$

(a). It's straightforward to compute the functional derivatives to get

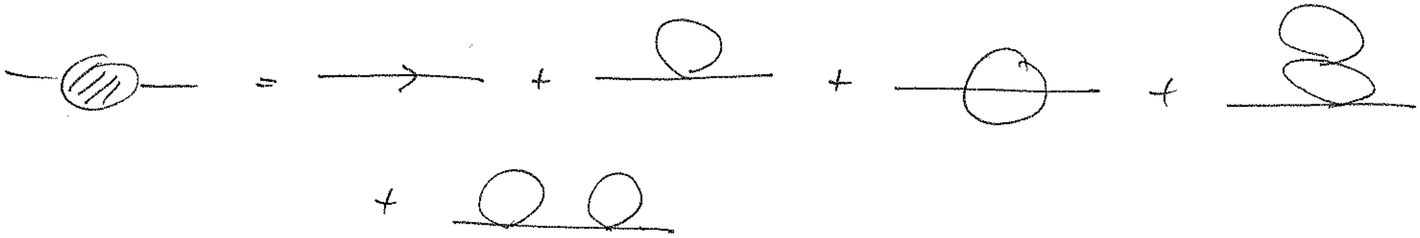
$$G^{(2)}(x_1,x_2) = D_F(x_1-x_2) - \frac{\lambda}{2}\int d^4y\, D_F(x_1-y) D_F(y-y) D_F(y-x_2)$$

$$+ \frac{\lambda^2}{6}\int d^4x d^4y\, D_F(x_1-x) D_F^3(x-y) D_F(y-x_2)$$

$$+ \frac{\lambda^2}{4}\int d^4x d^4y\, D_F(x_1-x) D_F^2(x-y) D_F(y-y) D_F(x-x_2)$$

$$+ \frac{\lambda^2}{4}\int d^4x d^4y\, D_F(x_1-x) D_F(x-x) D_F(x-y) D_F(y-y) D_F(y-x_2)$$

$$+ O(\lambda^3)$$



(b)

$$G^{(4)}(x_1,x_2,x_3,x_4) = -\lambda \int d^4x\, D_F(x_1-x)\, D_F(x_2-x)\, D_F(x_3-x)\, D_F(x_4-x)$$

$$+ \frac{\lambda^2}{2} \int d^4x\, d^4y \Big\{ D_F^2(x-y) \Big[D_F(x_1-x)\, D_F(x_2-x)\, D_F(x_3-x)\, D_F(x_4-y)$$

$$+ D_F(x_1-x)\, D_F(x_3-x)\, D_F(x_2-y)\, D_F(x_4-y)$$

$$+ D_F(x_1-x)\, D_F(x_4-x)\, D_F(x_2-y)\, D_F(x_3-y) \Big]$$

$$+ D_F(y-y)\, D_F(x-y) \Big[D_F(x_1-x)\, D_F(x_2-x)\, D_F(x_3-x)\, D_F(x_4-y)$$

$$+ \text{cyclic permutations} \Big] \Big\} + O(\lambda^3)$$