1.16

(a) The Probability for N molecules in V is given by the binomial distribution

$$W(N) = \frac{N_0!}{N!(N_0-N)!}\left(\frac{V}{V_0}\right)^N\left(1-\frac{V}{V_0}\right)^{N_0-N}$$

Thus $\bar{N} = N_0 P = N_0 \frac{V}{V_0}$

(b). $$\frac{\overline{(N-\bar{N})^2}}{\bar{N}^2} = \frac{\overline{N^2}-\bar{N}^2}{\bar{N}^2} = \frac{N_0\frac{V}{V_0}\left(1-\frac{V}{V_0}\right)}{\bar{N}^2} = \frac{1-\frac{V}{V_0}}{\bar{N}}$$

(c) If $V \ll V_0$, $\frac{\overline{(\Delta N)^2}}{\bar{N}^2} \approx \frac{1}{\bar{N}}$

(d) If $V \to V_0$, $\frac{\overline{(\Delta N)^2}}{\bar{N}^2} \to 0$

1.17 Since $0 \ll \frac{V}{V_0} \ll 1$ and since N_0 is large, we may use a Gaussian distribution.

$$P(N)dN = \frac{1}{\sqrt{2\pi\,\overline{\Delta N^2}}}\exp\left[-\frac{(N-\bar{N})^2}{2\,\overline{\Delta N^2}}\right]dN$$

1.22

(a) $\bar{x} = \sum_i^N \bar{s}_i$, but since $\bar{s}_i = \ell$,

$\bar{x} = \sum_i^N \ell = N\ell$

(b) $$\overline{(x-\bar{x})^2} = \sum_i^N \overline{(s_i-\ell)^2} + \sum_i\sum_j \overline{(s_i-\ell)}\,\overline{(s_j-\ell)}$$
$(i \neq j)$

Since $\overline{(s_i-\ell)} = \ell-\ell = 0$, the second term is 0 and

$$\overline{(x-\bar{x})^2} = \sum_i^N \sigma^2 = N\sigma^2.$$

2.4

(a) The number of ways that N spins can be arranged such that n_1 are parallel and n_2 are antiparallel to the field is

$$f(n_1) = \frac{N!}{n_1!\, n_2!} \qquad \text{and} \qquad \Omega(E) = f(n_1)\frac{\delta E}{2\mu H}$$

Since $E = -(n_1 - n_2)\mu H = -(2n_1 - N)\mu H$,

we have $n_1 = \frac{1}{2}(N - \frac{E}{\mu H})$, $n_2 = \frac{1}{2}(N + \frac{E}{2\mu H})$

Thus $$\Omega(E) = \frac{N!}{(\frac{N}{2} - \frac{E}{2\mu H})!\,(\frac{N}{2} + \frac{E}{2\mu H})!}\,\frac{\delta E}{2\mu H}$$

(b). Using Stirling's approximation, $\ln n! = (n \ln n) - n$,

we have $$\ln \frac{N!}{n_1!(N-n_1)!} = -n_1 \ln \frac{n_1}{N} + (n_1 - N)\ln(1 - \frac{n_1}{N})$$

$$\ln \Omega(E) = -\frac{1}{2}(N - \frac{E}{\mu H})\ln \frac{1}{2}(1 - \frac{E}{N\mu H}) - \frac{1}{2}(N + \frac{E}{\mu H})\ln \frac{1}{2}(1 + \frac{E}{N\mu H}) + \ln \frac{\delta E}{2\mu H}$$

(c) We expand $\ln f(n_1) = \ln \frac{N!}{n_1!(N-n_1)!}$ about the maximum, $\tilde{n}_1$

$$\ln f(n_1) = \ln f(\tilde{n}_1) + \frac{1}{2} B_2 \eta^2 \qquad (1)$$

where $\eta = n_1 - \tilde{n}_1$, $B_2 = \left[\frac{d^2 \ln f(n_1)}{dn_1^2}\right]_{\tilde{n}_1}$

From (1) $f(n_1) = f(\tilde{n}_1)\exp[-\frac{1}{2}|B_2|\eta^2]$ (2)

$f(\tilde{n}_1)$ is evaluated by noticing that the integral of $f(n_1)$ over all n_1 must equal the total number of permutations of spin directions, 2^N.

$$\int_{-\infty}^{\infty} f(\tilde{n}_1)\exp[-\frac{1}{2}|B_2|\eta^2]\,d\eta = 2^N$$

$$f(\tilde{n}_1) = 2^N\sqrt{\frac{|B_2|}{2\pi}}$$

$\tilde{n}_1$ is found from the maximum condition

$$\left|\frac{d\ln f(n_1)}{dn_1}\right|_{\tilde{n}_1} = 0 = -\ln \tilde{n}_1 + \ln(N - \tilde{n}_1), \text{ or } \tilde{n}_1 = \frac{N}{2}$$

and $$|B_2| = \left|\frac{d^2 \ln f(n_1)}{dn_1^2}\right|_{\tilde{n}_1} = \left|-\frac{1}{\tilde{n}_1} - \frac{1}{N - \tilde{n}_1}\right|_{\tilde{n}_1 = \frac{N}{2}} = \frac{4}{N}$$

Substituting these expressions into (2), we have

$$f(n_1) = \frac{2^N}{\sqrt{\pi \frac{N}{2}}}\exp[-\frac{2}{N}(n_1 - \frac{N}{2})^2]$$

Since $\Omega(E) = f(n_1)\frac{\delta E}{2\mu H}$, and $n_1 = \frac{1}{2}(N - \frac{E}{\mu H})$,

it follows that

$$\Omega(E) = \frac{2^N}{\sqrt{\pi \frac{N}{2}}} \exp\left[-\frac{2}{N}\left(\frac{E}{2\mu H}\right)^2\right] \frac{\delta E}{2\mu H}$$