Phyx 332-0 Statistical Mechanics

Midterm 04/27/11 Solutions.

Problem 1.

(a) $P(E) \propto \Omega^{(0)}(E)$.

using $\sum_E P(E) = 1 \Rightarrow P(E) = \dfrac{\Omega^{(0)}(E)}{\sum_E \Omega^{(0)}(E)}$

$$\Omega^{(0)}(E) = \Omega(E) \times \Omega'(E')$$
$$= \Omega(E) \times \Omega'(E^{(0)} - E).$$

(b) $$\log \Omega(E) = \log \Omega(\tilde{E}) + \left(\frac{\partial \log \Omega}{\partial E}\right)\eta + \frac{1}{2}\left(\frac{\partial^2 \log \Omega}{\partial E^2}\right)\eta^2 + \cdots$$

$$= \log \Omega(\tilde{E}) + \beta\eta - \frac{1}{2}\lambda\eta^2$$

where $\beta = \left(\dfrac{\partial \log \Omega}{\partial E}\right)$

$$\lambda = -\left(\frac{\partial^2 \log \Omega}{\partial E^2}\right) = -\left(\frac{\partial \beta}{\partial E}\right)$$

$$\log \Omega'(E') = \log \Omega'(\tilde{E}') + \beta'\eta' - \frac{1}{2}\lambda'\eta'^2$$

but $\eta' = E' - \tilde{E}' = -(E - \tilde{E}) = -\eta$ by energy conservation.

$$\Rightarrow \log \Omega'(E') = \log \Omega'(\tilde{E}') - \beta'\eta - \frac{1}{2}\lambda'\eta^2 + \cdots$$

(c) $P(E) = C \cdot \Omega(E)\Omega'(E')$ , $C$ = normalization constant.

$$\log P(E) = \log C + \log \Omega(E) + \log \Omega'(E')$$

$$= \log C + \log \Omega(\tilde{E}) + \beta\eta - \frac{1}{2}\lambda\eta^2$$

$$+ \log \Omega'(\tilde{E}') - \beta'\eta - \frac{1}{2}\lambda'\eta^2$$

$$= \log\left[C\Omega(\tilde{E})\Omega'(\tilde{E}')\right] + (\beta-\beta')\eta - \frac{1}{2}(\lambda+\lambda')\eta^2$$

$$= \log P(\tilde{E}) + (\beta-\beta')\eta - \frac{1}{2}(\lambda+\lambda')\eta^2.$$

at thermal equilibrium, $\beta = \beta'$

$$\Rightarrow \quad P(E) = P(\tilde{E})\, e^{-\frac{1}{2}(\lambda+\lambda')(E-\tilde{E})^2}$$

(d) $P(E)$ has the form of a Gaussian distribution.

Hence $\tilde{E} = \bar{E}$, and $\sigma = \Delta^* E = \frac{1}{\sqrt{\lambda_0}} \simeq \frac{1}{\sqrt{\lambda}}$ since $\lambda \gg \lambda'$.

Using $\Omega(E) \propto E^f$

$$\Rightarrow \log\Omega \propto f \log E \quad \Rightarrow \lambda = -\left(-\frac{f}{\tilde{E}^2}\right) = \frac{f}{\tilde{E}^2}$$

$$\Rightarrow \quad \frac{\Delta^* E}{\bar{E}} \simeq \frac{1}{\sqrt{f}}.$$

(e) $P(E)$ is maximum at $\tilde{E}$ $\Rightarrow$ 2nd derivative at $\tilde{E}$ is negative.

$$\therefore \lambda = -\frac{\partial^2 \log \Omega}{\partial E^2} = -\frac{\partial \beta}{\partial E} \geq 0$$

$$\Rightarrow \quad \frac{\partial T}{\partial E} \geq 0$$

(f). $S^{(0)} = k \log \Omega^{(0)} = k \log \frac{P(E)}{C}$ ; $C$ = normalization constant in (a).

$$(c) \Rightarrow \quad S^{(0)} = k \log \frac{P(\hat{E})}{C} - \frac{1}{2}\lambda_0 (E-\hat{E})^2.$$

$$= k \log \Omega^{(0)}(\hat{E}) - \frac{1}{2}\lambda_0 (E-\hat{E})^2.$$

Since $\lambda_0 \geq 0$, $S^{(0)}$ is maximum at $E = \hat{E}$.

Problem 2.

(a) $\Omega(E, V) = \rho(V)\,\chi(E)$

$$\beta = \frac{\partial \log \Omega}{\partial E} = \frac{\partial}{\partial E}\left(\log \rho(V) + \log \chi(E)\right)$$

$$= \frac{\partial}{\partial E} \log \chi(E).$$

$\therefore$ $\beta = \beta(E)$ is a function of $E$ but not $V$.

(b) $P(E, V) \propto \Omega^{(0)}(E, V) = \Omega(E, V)\,\Omega'(E', V')$.

$$\Rightarrow \log P(E, V) = \log C + \log \Omega(E, V) + \log \Omega'(E', V').$$

$$\log \Omega(E, V) = \log \Omega(\tilde{E}, \tilde{V}) + \left(\frac{\partial}{\partial E} \log \Omega(E, V)\right)(E - \tilde{E})$$

$$+ \left(\frac{\partial}{\partial V} \log \Omega(E, V)\right)(V - \tilde{V}) + \frac{1}{2}\left(\frac{\partial^2}{\partial E^2} \log \Omega\right)(E - \tilde{E})^2$$

$$+ \frac{1}{2}\left(\frac{\partial^2}{\partial E \partial V} \log \Omega\right)(E - \tilde{E})(V - \tilde{V}) + \frac{1}{2}\left(\frac{\partial^2}{\partial V^2} \log \Omega\right)(V - \tilde{V})^2$$

Since $\Omega = \rho(V)\,\chi(E)$

$$\log \Omega = \log \rho(V) + \log \chi(E)$$

$$\Rightarrow \frac{\partial^2}{\partial E \partial V} \log \Omega = 0$$

$$\therefore \log \Omega(E, V) = \log \Omega(\tilde{E}, \tilde{V}) + \beta(E - \tilde{E}) + \beta \bar{P}(V - \tilde{V})$$

$$- \frac{1}{2}\lambda(E - \tilde{E})^2 - \frac{1}{2}\delta(V - \tilde{V})^2,$$ and similarly for $\Omega'(E', V')$.

$$\Rightarrow \log P(E,V) = \log P(\tilde{E},\tilde{V}) + (\beta-\beta')(E-\tilde{E}) + (\beta\bar{p}-\beta'\bar{p}')(V-\tilde{V})$$
$$-\frac{1}{2}\lambda_0(E-\tilde{E})^2 - \frac{1}{2}\delta_0(V-\tilde{V})^2.$$

At equilibrium

$$\beta=\beta', \qquad \beta\bar{p} = \beta'\bar{p}'$$

$$\Rightarrow P(E,V) = P(\tilde{E},\tilde{V})\, e^{-\frac{1}{2}\lambda_0(E-\tilde{E})^2}\, e^{-\frac{1}{2}\delta_0(V-\tilde{V})^2}.$$

(c). $P(\tilde{E},\tilde{V})$ is maximum $\Rightarrow$ $\delta_0 \approx \delta \geq 0$

i.e., $-\beta \frac{\partial \bar{p}}{\partial V} \geq 0.$

Since $\beta \geq 0$ $\Rightarrow$ $\frac{\partial \bar{p}}{\partial V} \leq 0.$

(d). $\Omega$ = Volume in phase space occupied by states with energy between $E$ and $E+\delta E$.

i.e., $\Omega = \int_E^{E+\delta E} d^3x_1 \cdots d^3x_N\, d^3p_1 \cdots d^3p_N.$

Since $E$ is only consisted of kinetic energy, $E$ is independent of $x_i$

$$\Rightarrow \Omega = \int d^3x_1 \cdots d^3x_N \int_E^{E+\delta E} d^3p_1 \cdots d^3p_N$$
$$= V^N \cdot \int_E^{E+\delta E} d^3p_1 \cdots d^3p_N.$$

On the other hand,

$$2mE = \sum_i (p_{ix}^2 + p_{iy}^2 + p_{iz}^2)$$

is a sphere in the 3N dimensional phase space, and

$$\int_E^{E+\delta E} d^3p_1 \cdots d^3p_N$$ is the volume of a

spherical shell, which is proportional to

Radius$^{3N-1}$. But $R \propto \sqrt{E}$.

$$\Rightarrow \Omega \propto V^N E^{\frac{3N-1}{2}} \sim V^N E^{\frac{3}{2}N} \quad \text{for } N \gg 1.$$

(e). The same reasoning as above lead to

$\Omega_{2D} \propto a^N$ as before.

However, in the case $\int_E^{E+\delta E} d^2p_1 \cdots d^2p_N$ is a spherical shell in the 2N dimensional space.

$\therefore$ Volume $\propto R^{2N-1} \sim E^{(2N-1)/2}$

$$\Rightarrow \Omega_{2D} \propto a^N E^N.$$

(f) use the relation

$$\tau = \frac{1}{\beta} \frac{\partial \log \Omega}{\partial a}$$

$$\Rightarrow \quad \tau = \frac{1}{\beta} \frac{\partial}{\partial a} \log a^N = \frac{NkT}{a}$$

$$\Rightarrow \quad \tau a = NkT.$$