Phyx 332-0 Midterm Solutions

Problem 1

(a). The average energy of any quadratic degree of
(5pts) freedom is $\frac{1}{2}kT$.

(b) For a monoatomic gas, the quadratic degree of
(5pts) freedom is 3

$$\Rightarrow \bar{E} = 3 \times \frac{1}{2}kT. \qquad \text{total } E = N\bar{E}$$

(c) $C_V = \frac{\partial E}{\partial T} = \frac{3}{2}Nk$. N = # of particles.
(5pts)

(d) $\Delta U = Q + W$.
(5pts)

ΔU = change in the total energy
Q = heat absorbed by the system
W = work done on the system.

(e) First law $\Rightarrow$
(10 pts)

$$C_P = \left(\frac{\Delta U - W}{\Delta T}\right)_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P$$

where $\Delta W = -P\Delta V$

(f) (5 pts)

$C_p > C_V$ because additional heat is needed to compensate for the energy lost as work.

(g) (15 pts)

The heat capacity has the following characteristics:

$20\,K \lesssim T \lesssim 100\,K$ $\quad C_V = \frac{3}{2}R$

$100\,K \lesssim T \lesssim 1000\,K$ $\quad C_V = 5/2\,R$

$1000\,K \lesssim T$ $\quad C_V \to \frac{7}{2}R$.

The $\frac{3}{2}R$ at 100 K is due to 3 translational degree of freedom. (Kinetic energy in three different directions.)

$\frac{5}{2}R$ is due to two additional rotational degree of freedom for diatomic molecules. + kinetic energy

$\frac{7}{2}R$ is due to two vibrational degree of freedom (vibrational kinetic energy + vibrational potential energy) + $\frac{5}{2}R$

Problem 2

(a) (5pts) $P(E, A) \propto \Omega^{(0)}(E, A)$.

(b) (5pts) $\Omega^{(0)}(E, A) = \Omega(E, A) \times \Omega'(E', A')$.

(c) (5pts) $\log P$ is a much more slowly varying function than P itself.

Also $\log$ is a monotonically increasing function so that the maximum of P occurs at the same location as $\log P$

(d) (5pts) At equilibrium the probability is maximized

$$\Rightarrow \quad dP(E, A) = \frac{\partial P}{\partial E} dE + \frac{\partial P}{\partial A} dA = 0$$

i.e., the first total derivative vanishes.

(e) (10 pts) We should expect $T = T'$ and $\tau = \tau'$.
τ is the analog of volume in 3D.
The systems are balanced when the tension between them is the same.

(f) (10 pts) Let's look at $\log P = \log\Omega + \log\Omega'$

$$d\log P = 0 = \frac{\partial}{\partial E}\log\Omega \, \overset{dE}{\cancel{dA}} + \frac{\partial}{\partial A}\log\Omega\, dA + \frac{\partial}{\partial E'}\log\Omega' \cdot \frac{\partial E'}{\partial E}\, dE + \frac{\partial}{\partial A'}\log\Omega' \frac{\partial A'}{\partial A}\, dA$$

$$E + E' = E^{(0)} = \text{fixed} \Rightarrow \frac{\partial E'}{\partial E} = -1$$

$$A + A' = A^{(0)} = \text{fixed} \Rightarrow \frac{\partial A'}{\partial A} = -1$$

$$\therefore d\log P = 0 = \left(\frac{\partial}{\partial E}\log\Omega - \frac{\partial}{\partial E'}\log\Omega'\right) dE + \left(\frac{\partial}{\partial A}\log\Omega - \frac{\partial}{\partial A'}\log\Omega'\right) dA$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial E}\log\Omega = \frac{\partial}{\partial E'}\log\Omega' & \text{or} \quad \beta = \beta' \\ \frac{\partial}{\partial A}\log\Omega = \frac{\partial}{\partial A'}\log\Omega' & \text{or} \quad \frac{\partial}{\partial E}\log\Omega \cdot \frac{\partial E}{\partial A} = \frac{\partial}{\partial E'}\log\Omega' \cdot \frac{\partial E'}{\partial A'} \end{cases}$$

Since $\cancel{P = \frac{1}{\beta}\frac{\partial}{\partial V}\log\Omega}$. pressure $= \frac{1}{\beta}\frac{\partial}{\partial V}\log\Omega$.

$$\Rightarrow \tau = \frac{1}{\beta}\frac{\partial}{\partial A}\log\Omega = \frac{\partial E}{\partial A}$$

or, notice ~~σ~~ $\frac{\partial E}{\partial A}$ has the dimension of $\frac{\text{Force}}{\text{length}}$

$\Rightarrow$ the conditions are $T = T'$

$\tau = \tau'$

(g)
(10 pts)

$$\Omega(E, A) = \int_{E \le E' \le E+\delta E} d^2\vec{r}_1 \cdots d^2\vec{r}_N \, d^2\vec{p}_1 \cdots d^2\vec{p}_N$$

notice the position vectors are now 2-dimensional. as do the momentum vectors

For an ideal gas $E = \sum_i \frac{|\vec{p}_i|^2}{2m}$ is independent of position

$$\Rightarrow \Omega(E, A) = \left[\int d^2\vec{r}_1\right] \cdots \left[\int d^2\vec{r}_N\right] \times \int_{E \le E' \le E+\delta E} d^2\vec{p}_1 \cdots d^2\vec{p}_N$$

$$= A^N \cdot \chi(E)$$

$$\chi(E) = \int_{E \le E' \le E+\delta E} d^2\vec{p}_1 \cdots d^2\vec{p}_N.$$

From $\tau = \frac{1}{\beta} \frac{\partial}{\partial A} \log \Omega \Rightarrow \tau = \frac{1}{\beta} \frac{N}{A}$

or $\tau A = NkT$.