On Calculation of the Parameter Azwidth

Robert G. Wagner
Argonne National Laboratory, Argonne, Illinois 60439

July 27, 2005

Abstract

The Whipple/VERITAS analysis image parameter, Azwidth, usually has been defined as the signal-weighted RMS spread of photomultiplier tube signals perpendicular to the line joining the image centroid and the center of the field of view (FOV) of the camera. When a line depicting this quantity is shown on a diagram defining the Hillas parameters, it is typically drawn as a chord of the ellipse with major axis given by the Hillas Length parameter and minor axis given by the Hillas Width parameter. The chord drawn is one perpendicular to the image centroid/FOV-center line and the formula given for Azwidth corresponds to the length of the chord. If one takes as correct the stated definition of Azwidth, then neither depicting Azwidth as this chord nor calculating it from the usual stated formula are correct. In this note, I derive the formula for Azwidth as stated by its verbal definition and contrast it to the value that has typically been depicted and calculated.

1 Introduction

In analysis of Imaging Air Cherenkov Telescope (IACT) data for high energy gamma ray astronomy, the camera image is usually parametrized in terms of the Hillas Parameters [1]. These include the signal size, the location of the image centroid in two-dimensional camera coordinates, the distance of the centroid from the center of the field of view (FOV), a pair of second central moments of the image denoted as Length and Width, and the orientation of the image in the camera plane. As γ-ray shower images tend to have an elliptically shaped image, the Length and Width parameters are defined as the signal-weighted RMS spread of the photomultiplier tube (PMT) signals along the major and minor axes of the ellipse, respectively. The orientation of the image in the camera plane is then typically characterized by the angle, α, defined as the angle between the major axis of the ellipse and the line connecting the image centroid to the center of the FOV. The length of this line defines the Distance parameter and I refine to the line hereafter as the "Distance Line". Definitions and formulae for the Hillas parameters can be found in a variety of sources. For this note, I have made particular use of three doctoral theses: A.J. Rodgers (Whipple) [2], S.J. Fegan (Whipple/VERITAS) [3] and D. Berge (HESS) [4]; along with a review article by D.J. Fegan [5]. Other parameters of the image are defined in terms of the main set listed above. In particular, a parameter denoted Azwidth has been included in the Whipple/VERITAS analysis package and is defined, **in words**, as the signal-weighted RMS spread of PMT signals perpendicular to the Distance Line. When drawn on a figure illustrating the Hillas Parameters (see for example, Rodgers [2] and D.J. Fegan [5]), the Azwidth is shown as a chord of the Length-Width ellipse noted above. This chord passes through the image centroid and is perpendicular to the Distance Line (see figure 1). The value of Azwidth is then calculated as the length of this chord. I note that the current VERITAS analysis code supplied by the University of Leeds group uses this chord length as the value of Azwidth. Given the verbal definition of Azwidth, neither its depiction as a chord of the

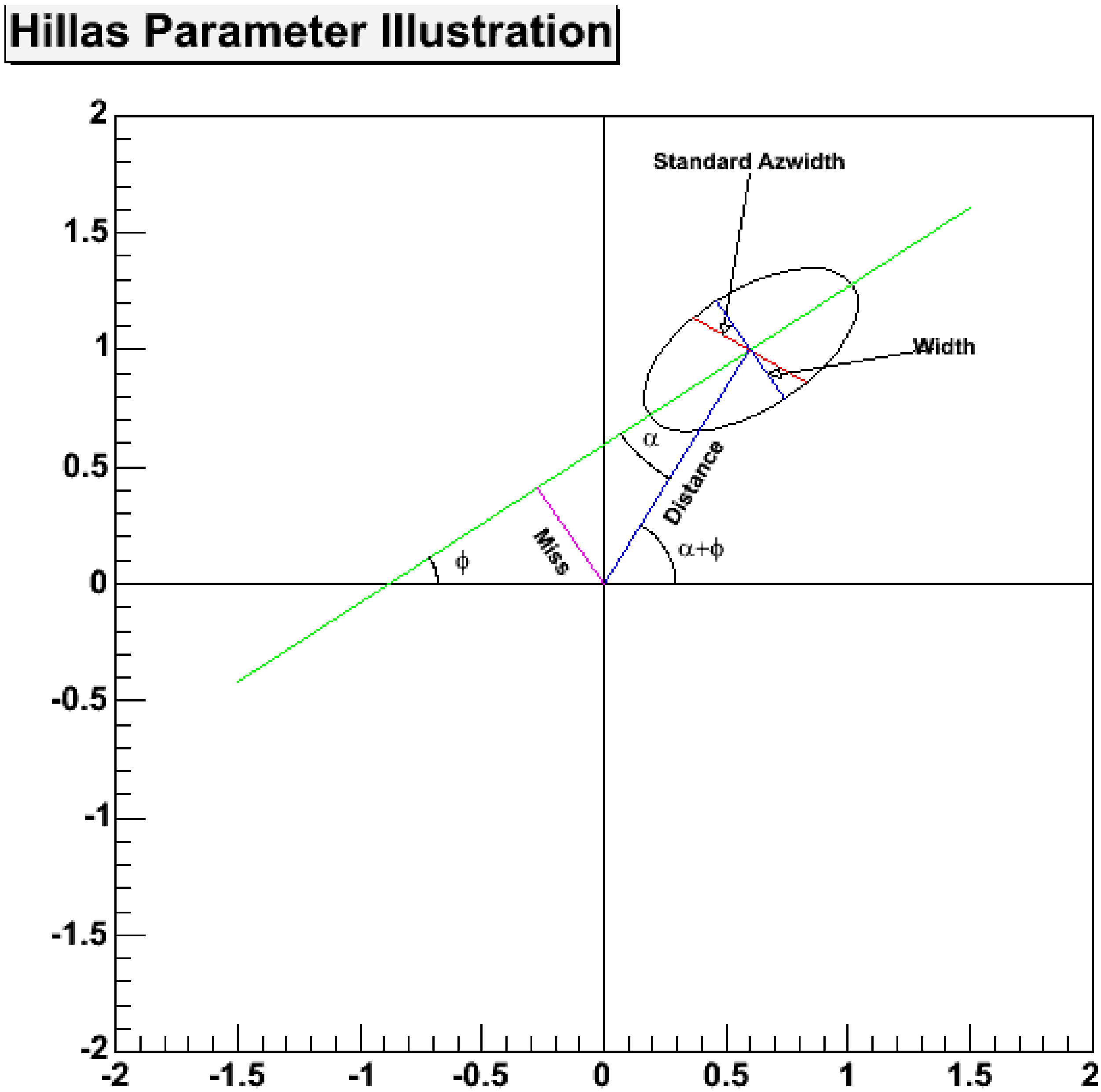

Figure 1: The standard depiction of the Hillas Parameters. The Azwidth parameter is usually shown as a chord of the ellipse whose major axis is the Hillas Length parameter and whose minor axis is the Hillas Width parameter. This depiction is not correct using the definition of Azwidth as the RMS spread perpendicular to the line shown above as Distance.

ellipse nor its calculation as the length of the chord are correct. The parameters, Length and Width, defining the ellipse are second central moments of the image calculated in a coordinate system oriented so as to diagonalize the second central moment tensor. The Azwidth parameter as defined is one of the diagonal components of the tensor calculated in a coordinate system rotated with respect to the Length-Width one. As such, its value is obtained by rotationally transforming the second central moment tensor to this rotated coordinate system. In the next section, I derive the formulae for the various second moment related parameters. In particular, Azwidth is obtained by performing the required transformation of the second moment tensor.

2 Formulae for Hillas Parameters

In what follows it will be useful to distinguish between the value of Azwidth calculated in the Whipple/VERITAS analysis package and the value derived as the RMS spread of PMT signals about the axis perpendicular to the Distance Line. I use the following terminology to make the distinction:

Standard Azwidth – The value of Azwidth calculated in the standard Whipple/VERITAS analysis package. This is the length of the chord of the Length-Width ellipse passing through the image centroid and perpendicular to the Distance Line.

RMS Azwidth – The value of Azwidth calculated as the signal-weighted RMS spread of PMT signals along an axis perpendicular to the Distance Line.

In order that the note be self-contained as much as possible, I give the formulae for the various camera image parameters. The image in the camera plane can be characterized in a two dimensional coordinate system using the coordinate notation x, y with the origin at the center of the camera FOV. The location of the i^{th} PMT in the camera is denoted as x_i, y_i and the size of its signal for a given event is denoted as s_i. We can then define the moments of the image about the origin:

Zeroth Moment: Size $\equiv S = \sum_i s_i$

First Moments:

$$x \text{ Centroid} \equiv < x >= \frac{1}{S} \sum_i x_i s_i$$

$$y \text{ Centroid} \equiv < y >= \frac{1}{S} \sum_i y_i s_i$$

Second Moments:

$$< x^2 >= \frac{1}{S} \sum_i x_i^2 s_i$$

$$< y^2 >= \frac{1}{S} \sum_i y_i^2 s_i$$

$$< xy >= \frac{1}{S} \sum_i x_i y_i s_i$$

I now define the second central moments in the x, y coordinate system. In keeping with the similarity of these to the mechanical moment of inertia tensor for a rigid body, I define a second central moment tensor:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

The diagonal elements, σ_{xx} and σ_{yy}, are sometimes also denoted as σ_{x^2} and σ_{y^2}. The elements are defined as

$$\sigma_{xx} \equiv < x^2 > - < x >^2$$

$$\sigma_{yy} \equiv < y^2 > - < y >^2$$

$$\sigma_{xy} \equiv < xy > - < x >< y >$$

The Hillas Parameters, Length and Width, represent the diagonal elements of the tensor when we rotate to a coordinate system, (x', y'), that diagonalizes the tensor, i.e.

$$\begin{pmatrix} \sigma_{x'x'} & 0 \\ 0 & \sigma_{y'y'} \end{pmatrix}$$

Denoting the rotation angle as ϕ (see figure 1), the rotation matrix is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The rotated tensor is obtained by multiplying on the left by the rotation matrix and on the right by its transpose:

$$\begin{pmatrix} \sigma_{x'x'} & 0 \\ 0 & \sigma_{y'y'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Performing the matrix multiplication and noting that ϕ was chosen so that $\sigma_{x'y'} = 0$, we obtain the constraint,

$$\sigma_{x'y'} = \sigma_{xy}(\cos^2\phi - \sin^2\phi) + (\sigma_{yy} - \sigma_{xx})\cos\phi\sin\phi = 0$$

I choose to express the solution in terms of $\tan\phi$ and the constraint equation becomes

$$\sigma_{xy}(1 - \tan^2\phi) + (\sigma_{yy} - \sigma_{xx})\tan\phi = 0 \tag{1}$$

Solving for $\tan\phi$, we have

$$\tan\phi = \frac{(\sigma_{yy} - \sigma_{xx}) + \sqrt{(\sigma_{yy} - \sigma_{xx})^2 + 4\sigma_{xy}^2}}{2\sigma_{xy}}.$$

We then obtain the following expressions for the Length and Width parameters:

$$(\text{Length})^2 \equiv \sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}\tan^2\phi + 2\sigma_{xy}\tan\phi}{1 + \tan^2\phi} \tag{2}$$

$$(\text{Width})^2 \equiv \sigma_{y'y'} = \frac{\sigma_{yy} + \sigma_{xx}\tan^2\phi - 2\sigma_{xy}\tan\phi}{1 + \tan^2\phi} \tag{3}$$

For any rotation the transformed tensor of second central moments is calculated in a similar manner. In particular, for Azwidth, we rotate to a coordinate system, (x'', y''), where the x'' axis is coincident with the Distance Line. Refering to figure 1, we see the rotation angle is $\phi+\alpha$. We get the transformed tensor via the multiplication,

$$\begin{pmatrix} \sigma_{x''x''} & \sigma_{x''y''} \\ \sigma_{x''y''} & \sigma_{y''y''} \end{pmatrix} = \begin{pmatrix} \cos(\phi+\alpha) & \sin(\phi+\alpha) \\ -\sin(\phi+\alpha) & \cos(\phi+\alpha) \end{pmatrix} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} \cos(\phi+\alpha) & -\sin(\phi+\alpha) \\ \sin(\phi+\alpha) & \cos(\phi+\alpha) \end{pmatrix}$$

The parameter, Azwidth, is then given by

$$(\text{Azwidth})^2 \equiv \sigma_{y''y''} = \frac{\sigma_{yy} + \sigma_{xx}\tan^2(\phi+\alpha) - 2\sigma_{xy}\tan(\phi+\alpha)}{1+\tan^2(\phi+\alpha)}.$$

Using

$$\tan(\phi+\alpha) = \frac{\tan\phi + \tan\alpha}{1-\tan\phi\tan\alpha},$$

we obtain

$$\begin{aligned} \sigma_{y''y''} \quad = \quad & \frac{1}{(1+\tan^2\phi)(1+\tan^2\alpha)} \times \{(\sigma_{yy} + \sigma_{xx}\tan^2\phi - 2\sigma_{xy}\tan\phi) + \\ & (\sigma_{yy}\tan^2 + \sigma_{xx} + 2\sigma_{xy}\tan\phi)\tan^2\alpha - \\ & 2\left[(\sigma_{yy}-\sigma_{xx})\tan\phi + \sigma_{xy}(1-\tan^2\phi)\right]\tan\alpha\} \end{aligned}$$

Note that the term in square brackets is equal to zero since it is simply the constraint for $\sigma_{x'y'}=0$. Bringing in the factor $(1+\tan^2\phi)$ from the denominator, we see the first term is simply $(\text{Width})^2$ and the second term is simply $(\text{Length})^2\tan^2\alpha$. So we find

$$(\text{Azwidth})^2 \equiv \sigma_{y''y''} = \frac{(\text{Width})^2 + (\text{Length})^2\tan^2\alpha}{1+\tan^2\alpha} \tag{4}$$

I note that one can also derive from the transformed second central moment tensor, the following expressions for $\sigma_{x''x''}$ and $\sigma_{x''y''}$:

$$\sigma_{x''x''} = \frac{(\text{Length})^2 + (\text{Width})^2\tan^2\alpha}{1+\tan^2\alpha}$$

$$\sigma_{x''y''} = \frac{\left[(\text{Width})^2 - (\text{Length})^2\right]\tan\alpha}{1+\tan^2\alpha}$$

By constrast, the Azwidth parameter in the Leeds analysis package for VERITAS is given in the *evndisp* code as

$$(\text{Azwidth})^2 = \left[1 + \frac{(\text{Length})^2 - (\text{Width})^2}{(\text{Width})^2 + (\text{Length})^2\cot^2\alpha}\right](\text{Width})^2$$

Or, in terms of $\tan\alpha$,

$$(\text{Azwidth})^2 = \frac{(\text{Length})^2(\text{Width})^2(1+\tan^2\alpha)}{(\text{Length})^2 + (\text{Width})^2\tan^2\alpha} \tag{5}$$

I note that the equation of the usual Hillas ellipse in the (x',y') coordinate system which is rotated from (x,y) by angle ϕ is

$$\frac{(x'-x'_c)^2}{(\text{Length})^2} + \frac{(y'-y'_c)^2}{(\text{Width})^2} = \frac{1}{4},$$

where (x'_c, y'_c) are the coordinates of the image centroid in the (x', y') system. In terms of the unrotated system,

$$x' - x'_c = (x- < x >) \cos\phi + (y- < y >) \sin\phi$$

$$y' - y'_c = -(x- < x >) \sin\phi + (y- < y >) \cos\phi.$$

The equation of the line perpendicular to the line shown as "Distance" in figure 1 is given by

$$y =< y > -(x- < x >) \cot(\phi + \alpha).$$

If we find the intersection of this line with the equation for the ellipse, we get two points on the perimeter of the ellipse that define the location of the Standard Azwidth line as given in the analysis code. It can then be shown that the distance between these points is the value given in equation 5 for Standard Azwidth, i.e. the length of the chord of the ellipse perpendicular to the Distance Line. In the next section, I contrast the difference between the two expressions for Azwidth.

3 Comparison of Azwidth Formulae

I begin by noting that the value for RMS Azwidth as given in equation 4 will always be larger than the value for Standard Azwidth in the VERITAS analysis package (equation 5). This is perhaps easiest to see by analogy with a moment of inertia tensor. The orientation that diagonalizes the tensor minimizes the moment of inertia for one of the axes. In our image case, the analogous quantity is the Width. Any re-orientation of the axes for the moment of inertia tensor increases the value of this diagonal element. Again, in our image case, the Width diagonal element gets transformed into the Azwidth value. It picks up contributions from PMTs at the ends of the perpendicular to Distance Line, i.e. PMTs at the ends of the major axis of the Length-Width ellipse. The Standard Azwidth value is derived simply from the Length-Width ellipse as a chord of the ellipse along this same axis. I illustrate the constrast in these two values by plotting the RMS Azwidth from the rotated tensor versus Standard Azwidth as used in the analysis code. This is shown for data from Run 620 for the VERITAS-1 telescope in figure 2a. This is an OFF source Crab run. No cuts have been made other than removing events having no tubes passing the cleaning procedure or having zero Width (which indicates Width below the zero tolerance threshold). As α approaches zero and the event becomes more γ-like, the two expressions for Azwidth will converge. This is illustrated in figure 2b where the following cuts have been applied:

- $0.4 <$ Distance < 1.0
- Width < 0.15
- Length < 0.3
- Length/Size < 0.0003
- $\alpha < 10°$.

One can see that the two Azwidth quantities are now essentially equal. While Azwidth has not been used in standard analyses, the value derived from the rotated moment tensor does have useful information. I speculate that it has perhaps been overlooked since the calculation that has been used doesn't provide much information beyond that given by Width and Length. Azwidth defined as the true RMS does combine the information of Length, Width, and α. As an illustration, I pick Event 86123 from Run 620 that has a large value of the RMS Azwidth compared to the Standard

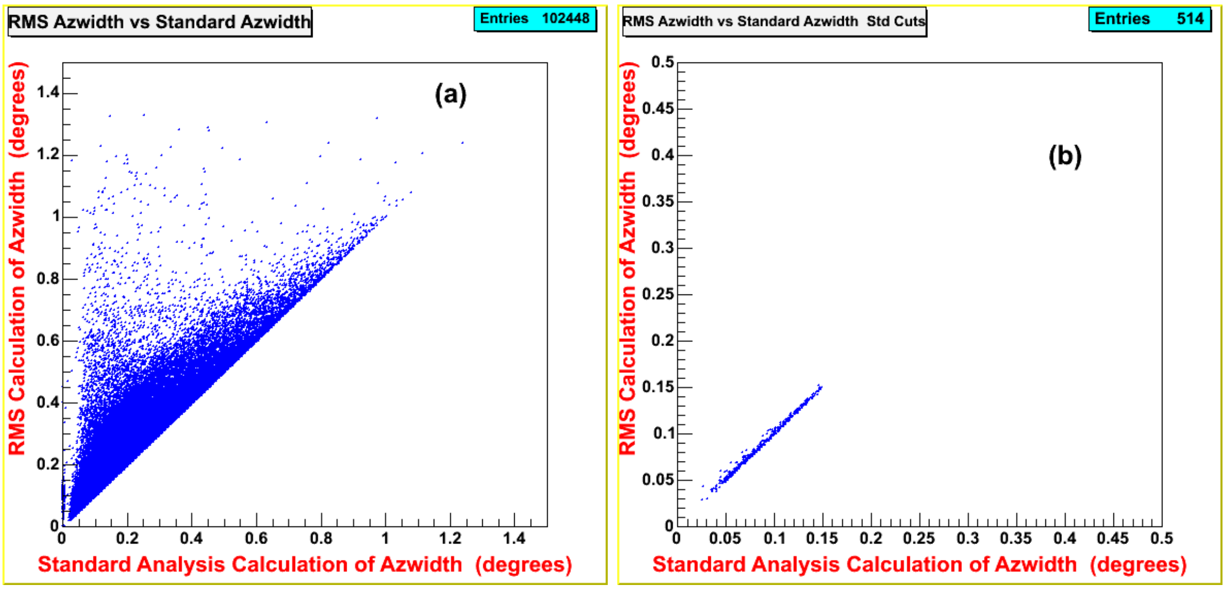

Figure 2: RMS Azwidth as given by equation 4 versus Standard Azwidth as calculated in the VERITAS analysis package (equation 5). Data are from Crab OFF source run 620 for the VERITAS-1 telescope. (a) No cuts applied other than removing events with zero Width or for which no tubes remained after standard cleaning. (b) Standard cuts applied as described in the text which enhance the sample in gamma ray like events.

Analysis Azwidth. The event has RMS Azwidth = 1.323 and Standard Azwidth = 0.149. Figure 3 shows the Length/Width ellipse for this event. We see that the Standard Azwidth is quite small due to the location of the image centroid with respect to the center of the FOV while RMS Azwidth is large due to both the large value of α and the large Length value. The figure also illustrates the true nature of the RMS spread definition for Azwidth. That is, we see that the long Length of the ellipse gives contributions to the second moment sum that are far from the centroid of image along the axis perpendicular to the Distance Line.

Once cuts have been made on Width, Length, and α, making Azwidth requirements probably adds little to the data purity as was seen in figure 2b. While RMS Azwidth may add little to the analysis, it might be desirable to convert to using the correct calculation of the variable according to the definition it has been given.

4 Summary

I've discussed in this note the discrepancy between the **word** definition of Azwidth and the value that is currently calculated in the Whipple/VERITAS analysis package. An expression has been derived for Azwidth that corresponds to the verbal definition, i.e. the signal-weighted RMS spread in PMT signals along an axis perpendicular to the line connecting the center of the camera FOV and the image centroid. The expression for Azwidth is obtained by transformation of the second moment tensor from its definition in the (x, y) frame to a frame rotated by angle $\phi + \alpha$. Here, α is the usual angle corresponding to the angle between the major axis of the image ellipse and the line connecting the image centroid and center of camera FOV; the Distance Line. ϕ is the angle the major axis of the ellipse makes with the x axis of the camera coordinate system. The

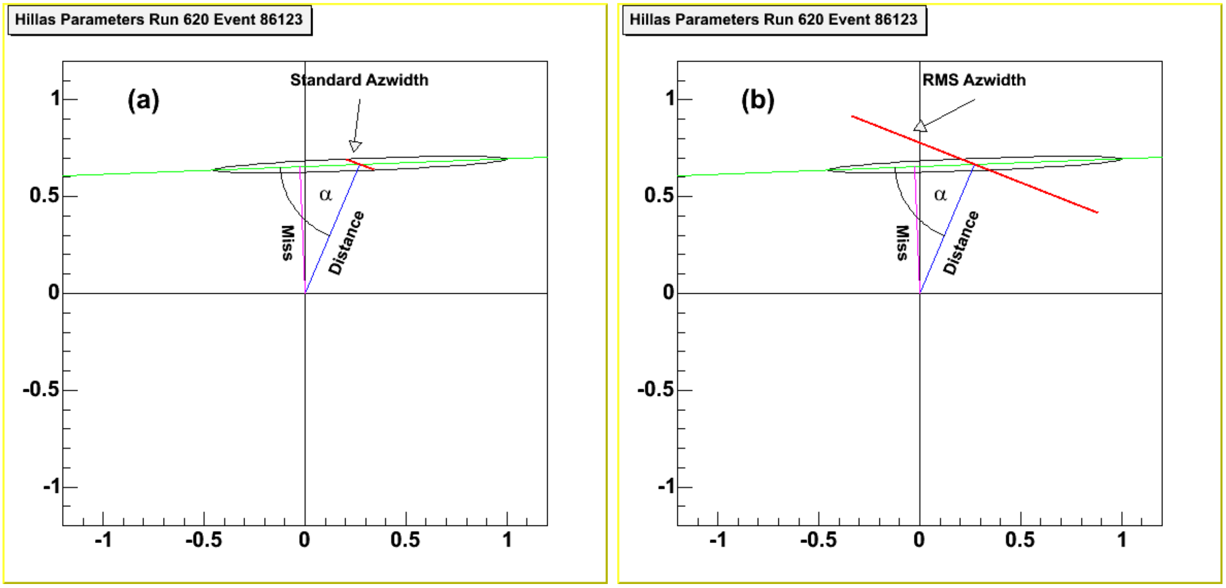

Figure 3: Comparison of Azwidth for Event 86123 from OFF source Crab Run 620 in which there is a large difference between (a) the Standard Analysis Azwidth and, (b) the RMS Azwidth. The location of the ellipse centroid causes the Standard Azwidth to be small while the large Length and α values cause the RMS Azwidth to be nearly equal to Length.

standard representation of Azwidth uses the length of the chord of the ellipse that is perpendicular to the Distance Line, and this can differ significantly from the value obtained as a true central second moment with respect to the axis perpendicular to the Distance Line. I would advocate either changing the Azwidth calculation in the VERITAS analysis package to the one derived in this note, or at least including this value in the list of variables calculated and kept for analysis output.

References

[1] A. M. Hillas, Proceedings of the 19$^{\text{th}}$ ICRC, vol. 3, La Jolla, USA, 445 (1985).

[2] A. J. Rodgers *"The Detection of Cosmic Gamma-Rays Using the Atmospheric Čerenkov Technique"* Doctoral thesis, University of Leeds (1997).

[3] S. Fegan *"A VHE Survey of Unidentified EGRET Sources"* Doctoral thesis, University of Arizona (2003).

[4] D. Berge *"Development of an Algorithm for the Shower Reconstruction with the H.E.S.S. Telescope"* Doctoral thesis, Humboldt-Universtät zu Berlin (2002).

[5] D.J. Fegan *"The Art and Power of Čerenkov Imaging*, Space Science Reviews **75**, 137 (1996).