Assignment 5: Rotations
Due: November 3, 2008

PHYX412-1 Fall 2008 : Quantum Mechanics I

Homework Assignment 5 : Rotations

1. Rotations in Four Dimensions

A strange alien being lives in a Universe with *four* spatial directions, which she writes in the analog of Cartesian coordinates as $\vec{x} = (x, y, z, w)$. She wants to understand the implications of her world for the quantum mechanics of angular momentum.

(Note: don't be confused with special relativity, these are four Euclidean dimensions; when our alien studies the version of relativity relevant for her Universe, she studies Lorentz *five* vectors).

A. Just as we did in the case of three dimensions, we begin by studying classical rotations of four dimensional space. Rotations leave the norm of a vector in 4-space invariant. Prove that this implies that we can represent a four-space rotation as a 4×4 orthogonal matrix acting on $\vec{x}$. Thus, her Universe's rotation group is $O(4)$.

B. A given rotation in 4-space mixes up coordinates of two of the axes, but leaves the coordinates of the other two axes invariant. So unlike our 3-space, we can't label a rotation by a single axis whose coordinate is invariant, we need a different labeling scheme which we will choose as the two axes which are transformed by the rotation. So for example the rotation $R^{(x,y)}(\theta)$ rotates the x and y coordinates into each other by an amount θ. Explain why this implies that there are *six* components of angular momentum in 4-space. Is angular momentum in a general dimensional space a vector?

C. Write down the six matrices which describe infinitesimal rotations between the cardinal axes to order ϵ^2 where ϵ is the very small rotation parameter.

D. Work out the commutators of all of the rotations of part **C** to order ϵ^2, and express the results in terms of the unit matrix and rotations by ϵ^2.

E. Now we apply our results to a quantum theory. We write an infinitesimal quantum mechanical rotation to order ϵ^2 as,

$$\hat{D}\left(R^{(i,j)}(\epsilon)\right) \equiv \hat{1} - \frac{i}{\hbar}\hat{J}_{(i,j)}\epsilon - \frac{1}{\hbar^2}\hat{J}^2_{(i,j)}\epsilon^2 + ...$$

where the $\hat{J}_{(i,j)}$ are the generators we are looking for and we can restrict $i > j$ without a loss of generality. Our $\hat{D}$'s must satisfy the same multiplication rules as the classical rotations did. Use that fact to work out the commutators of the $\hat{J}_{(i,j)}$ with each other.

F. In the four dimensional Universe, how many components of angular momentum can be measured simultaneously?

G. If we define a "total" angular momentum squared operator as,

$$\hat{J}^2 \equiv \sum_{i=1}^{4}\sum_{j>i}\hat{J}^2_{(i,j)}$$

Can we measure it simultaneously with any of the components $\hat{J}_{(i,j)}$?