INSTANTONS

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Contents

References

[1] S.Coleman, *Aspects of Symmetry*, Cambridge University Press, 1985. (**N.B.** This paper is based on Chapter 7 of this book.)

[2] M.E.Peskin and D.V.Schroeder, *An Introduction to Quantum Field Theory*, Westview Press, 1995.

I. INTRODUCTION

The central difference between classical and quantum physics can be put succinctly as the existence of a small parameter $\hbar \neq 0$, in quantum theory. One consequence of this fact is that, while the Lagrangian $\mathcal{L}$ is the natural functional in classical physics, the action functional $S = \mathcal{L}/\hbar$ is the corresponding relevant quantity in the quantum theory.

Let us take a specific example, that of ϕ^4 theory in 4 dimensions. The classical Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4. \tag{1}$$

Classically, the absolute value of λ does not matter since we can rescale the field and write the λ dependence as an overall factor. i.e. if we write $\phi' = \sqrt{\lambda}\phi$, we have

$$\mathcal{L} = \frac{1}{\lambda}\left(\frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2}m^2\phi'^2 - \phi'^4\right). \tag{2}$$

In the quantum theory, we have to look at $\mathcal{L}/\hbar$. This implies that the relevant parameter here is the dimensionless $\lambda\hbar$. Semiclassical small $\hbar$ approximation is equivalent to small λ approximation, and we can use these parameters interchangeably.

Theories with small (weak) coupling are studied typically using perturbation theory, order by order in the coupling parameter. But this might not capture all the interesting physics. To see this, consider the tunnelling amplitude through a potential barrier $V(x)$ in the semiclassical approximation

$$|T(E)| = \exp\left\{-\frac{1}{\hbar}\int_{x_1}^{x_2} dx\, [2(V-E)]^{1/2}\right\}[1 + O(\hbar)], \tag{3}$$

where x_1 and x_2 are the classical turning points at energy E. We see that the leading behavior of $\exp(-1/\hbar)$ cannot be captured by any small $\hbar$ perturbation expansion.

To study the analogs of similar non-perturbative phenomena in quantum field theory, we use the concept of instantons.

II. INSTANTONS IN QUANTUM MECHANICS

Our final goal is to understand instantons in a field-theoretic setting. But since we are starting with a quantum mechanical system, we note at the outset, that the position of

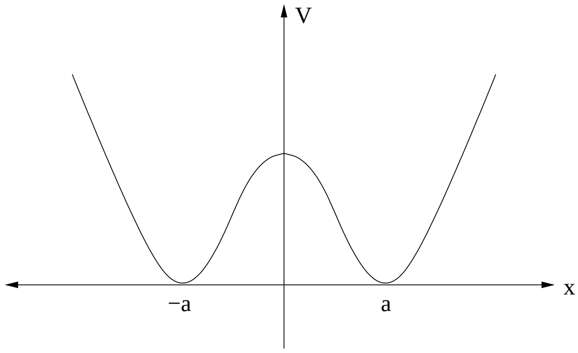

FIG. 1: Symmetric Double Well Potential

a particle in quantum mechanics plays the role of a quantum field in field theory. Hence quantum mechanics is just field theory in $0+1$ spacetime dimensions. Broadly speaking, instantons arise in quantum mechanics (for that matter, in field theory too) whenever there exists more than one classical ground state and quantum tunnelling can occur between these ground states. A typical example of such a system is the symmetric double well potential shown in Fig.1.

A starting point for studying such systems is to analyze the transition amplitude between an initial and final state (which can be assumed to be a position operator eigenstate) in the large time limit. The transition amplitude between the initial position x_i at time $t_i = -T/2$ and the final position x_f at time $t_f = T/2$ can be written as

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = N \int [dx] e^{-S/\hbar}. \tag{4}$$

The L.H.S. is in the operator notation while the R.H.S. is in the Feynman path-integral formalism. N is a normalization constant. We will be working with a Wick-rotated Euclidean time throughout this paper.

To interpret this, we insert a complete set of energy eigenstates into the L.H.S. as

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = \sum_n e^{-E_n T/\hbar} \langle x_f | n \rangle \langle n | x_i \rangle \text{ where } H|n\rangle = E_n |n\rangle. \tag{5}$$

Evaluating the above transition amplitude in the large T limit will give us the lowest energy eigenvalue and the corresponding eigenstate.

Considering the R.H.S, the action S in the exponential is given by

$$S = \int_{-T/2}^{T/2} dt \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + V \right] \tag{6}$$

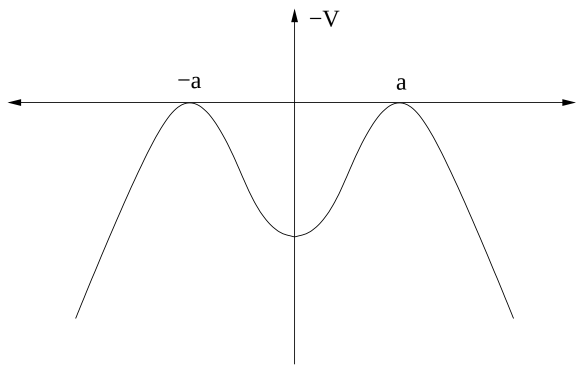

FIG. 2: Inverted Symmetric Double Well Potential

and $[dx]$ denotes sum over all paths obeying the boundary conditions $x(-T/2) = x_i$ and $x(T/2) = x_f$.

In the large T limit, the path-integral will be dominated by the stationary points of the action and evaluating it about these stationary points will tell us about the low-lying levels and their corresponding eigenstates. The stationary solutions of the action are given by

$$\frac{\delta S}{\delta \overline{x}} = 0 = -\frac{d^2\overline{x}}{dt^2} + V'(\overline{x}) \tag{7}$$

This condition can be thought of as the classical equation of motion of a particle of unit mass moving in a potential $-V(\overline{x})$ and hence

$$E = \frac{1}{2}\left(\frac{d\overline{x}}{dt}\right)^2 - V(\overline{x}) \tag{8}$$

is a constant of motion. Applying this to the inverted double well (Fig.2), we see that two obvious classical solutions obeying the boundary condition $x_i = \pm a$ and $x_f = \pm a$ are those cases in which the particle just sits on top of one of the hills. The other non-trivial classical solution is when the particle starts from one hill at $t_i = -T/2$ and reaches the top of the other hill at $t_f = T/2$. Since we are looking at the $T \to \infty$ limit, this solution occurs at $E = 0$. Hence from Eq.(8),

$$\frac{d\overline{x}}{dt} = \sqrt{2V(\overline{x})}, \tag{9}$$

Integrating this, we get

$$t = t_0 + \int_0^{\overline{x}} dx' \left[\sqrt{2V(\overline{x})}\right]^{-1} \tag{10}$$

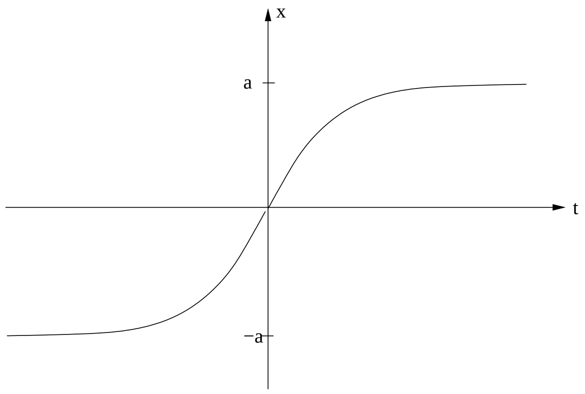

FIG. 3: Double Well Instanton

The solution looks (refer Fig.3) very similar to static soliton solutions of non-linear differential equations. In our case, the solutions are functions of time, hence the name 'Instantons'.

Our goal now is to calculate the energy and eigenstate corresponding to this non-trivial 'instanton' configuration. Here onwards, all the calculations in this section are very similar to the background field method (refer [2], Sec. 13.3 and 16.6). We integrate over all leading order fluctuations about a particular stationary solution. These fluctuations will be denoted by $x_n(t)$ and we write

$$x(t) = \overline{x}(t) + \sum_n c_n x_n(t), \tag{11}$$

where the $x_n(t)$ are fluctuations satisfying the boundary conditions $x_n(-T/2) = 0$ and $x_n(T/2) = 0$ and are orthogonal in the sense of

$$\int_{-T/2}^{T/2} dt x_n(t) x_m(t) = \delta_{nm}. \tag{12}$$

This integration is just a Gaussian integral and gives the usual square root determinant in the denominator:

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = N e^{-S(\overline{x})/\hbar} \det\left(-\partial_t^2 + V''(\overline{x})\right)^{-1/2} [1 + O(\hbar)] \tag{13}$$

This determinant can be evaluated by using a suitable regulator scheme for simple cases. Let us first calculate it for the 'uninteresting' case when the particle just sits on top of one of the hills in Fig. 2. In this case, let $V''(\overline{x} = \pm a) = \omega^2$. Evaluating the determinant in this case, gives in the large T limit,

$$N\det\left(-\partial_t^2+\omega^2\right)^{-1/2}=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}e^{-\omega T/2}. \tag{14}$$

Hence using the observation below Eq.(5), we see that the ground state energy is

$$E_0=\frac{1}{2}\omega\hbar\left[1+O(\hbar)\right] \tag{15}$$

and the corresponding probability for staying at the bottom of the wells is given by

$$|\langle x=\pm a|n=0\rangle|^2=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}\left[1+O(\hbar)\right] \tag{16}$$

This agrees with our intuitive understanding that in the absence of tunnelling, the particle just sees a harmonic potential, with the ground state resembling the harmonic oscillator ground state.

Now let us consider the case of the instanton solution. We first calculate $S(\overline{x})$,

$$S(\overline{x})=\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2+V\right]=\int_{-a}^{a}dx(2V)^{1/2}. \tag{17}$$

For large t, x approaches $\pm a$. Hence

$$\frac{dx}{dt}=\omega(a-x) \tag{18}$$

$$\Rightarrow (a-x)\propto e^{-\omega t} \tag{19}$$

Thus we see that the instanton is localized in time with width of the order $1/\omega$. This tells us that in the large T limit, we should consider not just the one instanton solution, but sum over all configurations with arbitrary number of well separated instantons and anti-instantons (which are just the negative of the instanton solution). Let us study the n instanton case. Since all the instantons are widely separated in time, the action is just nS_0 where S_0 is the one instanton action Eq. (17). Let the centers of these n instantons be at $t_1, t_2 \ldots t_n$ where

$$\frac{T}{2}>t_1>t_2\ldots>t_n>-\frac{T}{2} \tag{20}$$

To evaluate the determinant in the n instanton case, we note that most of the time, the expectation value of x is near $\pm a$ and it resembles the harmonic oscillator state. Thus the leading behavior is going to be just the harmonic oscillator answer with an n instanton multiplicative factor K^n:

$$\left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} K^n. \tag{21}$$

We can determine K later by considering the one-instanton case. Since the location of the center of each of the n instantons is arbitrary, we have to integrate over these positions,

$$\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{t_1} dt_2 \ldots \int_{-T/2}^{t_{n-1}} dt_n = \frac{T^n}{n!} \tag{22}$$

Each instanton has to be followed by an anti-instanton and vice-versa. Consequently, depending on whether we are calculating $\langle \pm a|e^{-HT/\hbar}|\pm a\rangle$ or $\langle \pm a|e^{-HT/\hbar}|\mp a\rangle$, we have to sum over only even or odd n respectively.

$$\langle a|e^{-HT/\hbar}|\pm a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{\text{even/odd n}} \frac{\left(Ke^{-S_0/\hbar}T\right)^n}{n!}\left[1+O(\hbar)\right], \tag{23}$$

$$= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}\frac{1}{2}\left[\exp\left(KTe^{-S_0/\hbar}\right) \pm \exp\left(-KTe^{-S_0/\hbar}\right)\right]. \tag{24}$$

Thus we see that there are two low lying states $|+\rangle$ and $|-\rangle$ whose energy difference is given by $2\hbar K e^{-S_0/\hbar}$ and the probability of being in these states, to the lowest order, is the same as being in the harmonic oscillator ground state. The energy splitting can be derived directly using WKB approximation, but our understanding of how this came about using the instanton picture, is easier to generalize to quantum field theories.

K can be calculated by considering the one instanton contribution to Eq. (23) and equating it to a direct calculation. The direct calculation gives

$$\langle a|e^{-HT/\hbar}|-a\rangle_{\text{one inst.}} = NT\left(\frac{S_0}{2\pi\hbar}\right)^{1/2} e^{-S_0\hbar}\left(-\partial_t^2+V''(\overline{x})\right)^{-1/2}$$
$$\times \det{}'\left(-\partial_t^2+V''(\overline{x})\right)^{-1/2} \tag{25}$$

where the prime above the determinant indicates that we do not consider the zero eigenvalue which arises as a result of the translational degree of freedom in choosing the center of the instanton. Comparing this to the one instanton contribution to Eq. (23), we deduce

$$K = \left(\frac{S_0}{2\pi\hbar}\right)^{1/2}\left|\frac{\det\left(-\partial_t^2+\omega^2\right)}{\det'\left(-\partial_t^2+V''(\overline{x})\right)}\right|^{1/2} \tag{26}$$

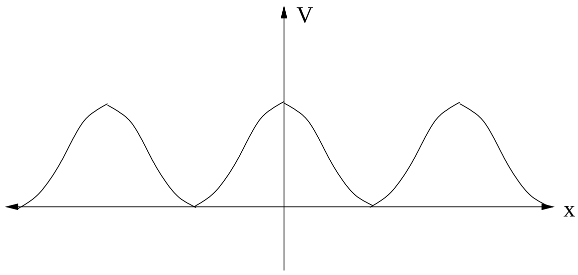

FIG. 4: Periodic Potential

Instantons in Periodic Potentials:

The double well instanton analysis can be readily generalized to the periodic potential case shown in Fig.4, where the minima are assumed to be located at unit intervals. Here the number of instantons and anti-instantons are independent unlike the double well case, where each instanton had to be followed by an anti-instanton. The only restriction is that the difference between the number of instantons and anti-instantons should be equal to the distance between the points about which we are calculating the transition amplitude. Hence the analog of Eq.(23), in this case is

$$\langle n_f|e^{-HT/\hbar}|n_i\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n,\overline{n}=0}^{\infty} \frac{\left(Ke^{-S_0/\hbar}T\right)^{n+\overline{n}}}{n!\overline{n}!}\, \delta_{n-\overline{n},n_f-n_i}. \tag{27}$$

We can write the Kronecker-Delta function as a Fourier transform,

$$\delta_{n-\overline{n},n_f-n_i} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(n-\overline{n}-n_f+n_i)}. \tag{28}$$

Now doing the sum over n and $\overline{n}$ gives

$$\langle n_f|e^{-HT/\hbar}|n_i\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(n_f-n_i)} \exp\left(2KT\cos\theta e^{-S_0/\hbar}\right) \tag{29}$$

Thus we find a continuum of eigenstates $|\theta\rangle$ labelled by the angle 'θ', with energy

$$E(\theta) = \frac{1}{2}\hbar\omega + 2\hbar K\cos\theta e^{-S_0/\hbar} \tag{30}$$

and the wavefunction amplitude given by

$$\langle\theta|x\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \frac{1}{2\pi} e^{ix\theta} \tag{31}$$

We see that our instanton-based analysis has yielded just the usual Bloch states in a periodic potential and the parameter 'θ' is nothing but the quasi-momentum in the first Brillouin zone.

III. FINITE ACTION CONFIGURATIONS IN GAUGE THEORIES

We summarize our gauge theory conventions in Appendix A. To begin our analysis of instantons in gauge theories, it is clear that we have to identify field configurations for which the action is finite. In addition, for these configurations to represent physical vacua, they must be gauge invariant. For finiteness of the action, the field strength has to fall off atleast as fast as $(1/r^3)$ at infinity. A naive conclusion from this will be that the gauge potential will have to go as $1/r^2$. But this is not true, what we require is that

$$A_\mu \sim g\partial_\mu g^{-1} + O\left(\frac{1}{r^2}\right), \tag{32}$$

where $g(x)$ can be a function of the four dimensional angular variables only. This means that every finite field configuration is associated with a mapping of the hypersphere, S^3 to the gauge group, G. But each such mapping is not gauge invariant. Under a gauge transformation,

$$A_\mu \rightarrow hA_\mu h^{-1} + h\partial_\mu h^{-1}. \tag{33}$$

Thus

$$g \rightarrow hg + O\left(\frac{1}{r^2}\right). \tag{34}$$

Now if we could reduce g to identity by choosing h to be g^{-1}, then we could eliminate g from Eq. (32). But this is not possible in general, because we require $h(x)$ to be continuous, not just over the hypersphere S^3 at infinity, but it has to be continuous over the whole four dimensional space. In particular, we should be able to go continuously from $h(x = 0)$ to infinity. At the origin, $h(x)$ has to be constant, independent of angular variables, for it to be well defined. We note that all constant gauge transformations are homotopic to the identity transformation. Hence we conclude that gauge invariant finite field configurations are those for which h at infinity is homotopic to the identity. This means that our first task is to identify the homotopy classes for mapping S^3 to the gauge group G.

We will study two specific examples and later use a theorem from Lie group theory to generalize this to arbitrary gauge groups.

First we consider Abelian $U(1)$ gauge theory in two-dimensional Euclidean space. Each element of $U(1)$ can be represented by $e^{i\theta}$. This is isomorphic to the circle S^1. Our group representation maps the spatial circle S^1 onto the U(1) circle S^1. It is clear that the homotopy classes are given by the number of times one circle wraps around the other. Hence the elements of all homotopy classes can be represented by

$$g(\theta) = e^{i\nu\theta} \tag{35}$$

where ν is an integer and $0 \leq \theta < 2\pi$. Hence each ν represents one homotopy class. We can write ν in terms of any arbitrary member of the corresponding homotopy class as

$$\nu = \frac{i}{2\pi} \int_0^{2\pi} d\theta g \frac{dg^{-1}}{d\theta} \tag{36}$$

Also it is easy to see that if

$$g(\theta) = g_1(\theta) g_2(\theta) \text{ then } \nu = \nu_1 + \nu_2. \tag{37}$$

Now if we define

$$G_\mu = \frac{i}{2\pi} \epsilon_{\mu\nu} A_\nu, \tag{38}$$

then using Eq.(32), we can write

$$\nu = \lim_{r \to \infty} \int_0^{2\pi} r d\theta \hat{r}_\mu G_\mu, \tag{39}$$

where $\hat{r}_\mu$ is the unit radial vector. Use Gauss's theorem and the definition of field strength to write

$$\nu = \int d^2x \partial_\mu G_\mu = \frac{i}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu} \tag{40}$$

This completes our task of identifying the homotopy classes in this problem and writing an expression for it in terms of the physical field strength.

Next we consider the case of the non-abelian gauge group, $SU(2)$ in four dimensional Euclidean space. Again, $SU(2)$ is topologically identical to S^3. Hence, here we consider the mapping of S^3 onto S^3. This homotopy group is again isomorphic to Z, the group of integers under addition. Proceeding very similar to the previous case, (Since the details of this calculation are not very instructive or essential for further discussion, I omit them here, they can be found in [1]), we can write the winding number in terms of field strength as

$$32\pi^2 \nu = \int d^4x (F, \tilde{F}), \tag{41}$$

where $\tilde{F}$ is the dual field strength given by $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$.

We can use this result for any arbitrary non-abelian gauge group since there is a theorem in group theory called Bott's theorem which states that the mapping of S^3 onto any group G can be continuously deformed into an $SU(2)$ subgroup of G.

IV. Θ VACUA

In the previous section, we identified that each finite action gauge invariant field configuration was in one-to-one correspondence with the winding number 'ν'. Let us put the system in a box of space-time volume VT. Now we want to calculate the energy of these finite action configurations. For this, first we have to make sure we are in an energy eigenstate. We can identify any state as a stationary state if the contribution from that state to the path integral, F can be time-factorized. i.e.

$$F(V, T_1 + T_2) = F(V, T_1)F(V, T_2) \tag{42}$$

Let the contribution to the path integral from a configuration with definite winding number n be $F(V, T, n)$. From Eq.(41), we see that these quantities obey the law

$$F(V, T_1 + T_2, n) = \sum_{n_1, n_2} F(V, T_1, n_1)F(V, T_2, n_2)\delta_{n_1+n_2, n} \tag{43}$$

The restriction on $n_1 + n_2$ arises because of the composition property in Eq.(37). We see that these states with definite winding number are not energy eigenstates since they do not obey the corresponding simple time-factorization property. This can be easily remedied if we observe that the above composition law for the winding states is a convolution in the winding number space. Hence if we go to the Fourier transform space, they will time factorize,

$$F(V, T, \theta) = \sum_n e^{in\theta}F(V, T, n) \tag{44}$$

$$\Rightarrow F(V, T_1 + T_2, \theta) = F(V, T_1, \theta)F(V, T_2, \theta) \tag{45}$$

We denote these energy eigenstates as $|\theta\rangle$ (these are the famous/infamous 'θ' vacua). We can write in this basis, ($\hbar = 1$)

$$F(V, T, \theta) \propto \langle\theta|e^{-HT}|\theta\rangle. \tag{46}$$

$$= N \int [dA]e^{-S}e^{i\nu\theta}. \tag{47}$$

All this analysis bore a remarkable similarity to the periodic potential case in quantum mechanics. This can be understood if we think of the winding numbers as labelling the points on a one-dimensional lattice and the θ vacua as nothing but the Bloch eigenstates.

The expression for the ground state energy can be written down in analogy with the periodic potential case. Here we have to write spacetime volume 'VT' instead of 'T' since we are working in a $3+1$ dimensional field theory. Hence the path integral contribution from a single θ eigenstate is

$$\langle\theta|e^{-HT}|\theta\rangle \propto \sum_{n,\overline{n}=0}^{\infty} \frac{\left(Ke^{-S_0/\hbar}VT\right)^{n+\overline{n}} e^{i(n-\overline{n})\theta}}{n!\overline{n}!} \tag{48}$$

$$= \exp\left(2KVTe^{-S_0}\cos\theta\right), \tag{49}$$

where the factor 'K' comes from the usual determinant got by integrating out the fluctuations around the classical vacuum. Hence the ground state energy density is given by

$$\frac{E(\theta)}{V} = -2K\cos\theta e^{-S_0}. \tag{50}$$

We can calculate expectation value of other operators w.r.t this 'θ' vacuum. As an example,

$$\langle\theta|(F(x),\tilde{F(x)})|\theta\rangle = \frac{1}{VT}\int d^4x\langle\theta|(F,\tilde{F})|\theta\rangle \quad \text{(now use Eq.(41))} \tag{51}$$

$$= \frac{32\pi^2\int[dA]\nu e^{-S}e^{i\nu\theta}}{VT\int[dA]e^{-S}e^{i\nu\theta}} \tag{52}$$

$$= -\frac{32\pi^2 i}{VT}\frac{d}{d\theta}\ln\left(\int[dA]e^{-S}e^{i\nu\theta}\right) \quad \text{(now use Eq.(47) and Eq.(48))} \tag{53}$$

$$= -64\pi^2 iKe^{-S_0}\sin\theta. \tag{54}$$

This completes our analysis of the θ vacua. We note that these vacua are indeed physically different, since physical quantities such as energy density and expectation value of various operators depend on the particular θ vacuum we are in.

V. CONFINEMENT IN 2 DIMENSIONS

Here, we study a toy model where non-perturbative physics (in our case, instantons) completely changes the perturbation theory based particle spectrum.

Consider the Abelian Higgs model in two dimensions:

$$\mathcal{L} = \frac{1}{4e^2}(F,F) + D_\mu\psi^* D_\mu\psi + \frac{\lambda}{4}(\psi^*\psi)^2 + \frac{\mu^2}{2}\psi^*\psi. \tag{55}$$

Here ψ is a complex scalar field coupled to an Abelian gauge field.

If $\mu^2 > 0$, then in 4 space-time dimensions, the spectrum is simple, consisting of a charged meson, its anti-particle and the massless vector gauge boson. In two spacetime dimensions, there is no photon as there are no available transverse directions for a massless gauge field. Hence the Coulomb force say, between two external charges, will be independent of distance. This means we cannot really separate a meson and its anti-particle and they are 'confined'. The spectrum is similar to a positronium bound state except that the meson bound state is stable as it cannot annihilate into photons.

The more interesting part comes when $\mu^2 < 0$. Normally in higher dimensions, we will expect the Anderson-Higgs mechanism to occur due to spontaneous symmetry breaking. The gauge field will eat the Goldstone boson and become massive. But there is a famous theorem due to Mermin-Wagner and Coleman which states that no spontaneous symmetry breaking can occur in two dimensions. How do we reconcile these opposing statements? Instantons save us from this quandary and we will see that we can retain the original picture of 2D confinement which occurs for $\mu^2 > 0$. But we will have an exponential dependence of the long range force on $\hbar$ unlike the latter case where the force does not depend on $\hbar$.

First we set the minimum of the action to zero. We add a constant to write it as

$$\mathcal{L} = \frac{1}{4e^2}(F,F) + |D_\mu\psi|^2 + \frac{\lambda}{4}\left(|\psi|^2 - a^2\right)^2. \tag{56}$$

For the third term to remain finite at $r = \infty$, $|\psi|$ has to approach a. But there is no restriction on the phase of ψ. Hence

$$\lim_{r\to\infty}\psi(r,\theta) = a\, g(\theta), \tag{57}$$

where g is complex with $|g| = 1$ and g is an element of $U(1)$. Using this result and demanding that the second term remain finite as well when $r \to \infty$, we get

$$A_\mu = g\partial_\mu g^{-1} + O\left(\frac{1}{r^2}\right) \tag{58}$$

This condition is exactly similar to Eq.(32). Hence we can use the two dimension $U(1)$ gauge theory results which we derived in that section. In particular, finite energy configurations are classified by the winding number ν. From Eq.(40) and using Stokes' theorem, we can write

$$\nu = \frac{i}{4\pi} \int d^2 x \epsilon_{\mu\nu} F_{\mu\nu} = \frac{i}{2\pi} \oint A_\mu dx_\mu \tag{59}$$

Just as before, we will have a set of θ vacua, with energy densities given by

$$\frac{E(\theta)}{L} = -2K \cos\theta e^{-S_0}. \tag{60}$$

Similar to Eq.(51), we can write

$$\langle\theta|\epsilon_{\mu\nu} F_{\mu\nu}|\theta\rangle = 8\pi i K e^{-S_0} \sin\theta. \tag{61}$$

Next we move on to study the Coulomb force between external test charges. This is done by introducing two static charges of magnitude q and opposite sign separated by a distance L'. Then we calculate the change in vacuum energy Δ, due to these external charges. For this, we study the finite loop comparator involving a Wilson loop integral

$$W = \exp\left(-\frac{q}{e} \oint A_\mu dx_\mu\right), \tag{62}$$

where the loop is a rectangle of spacetime area $L'T'$. The term in the exponential is just the energy density from all the configurations within the loop. Then the shift in vacuum energy for the θ vacuum is given by

$$\Delta = -\lim_{T' \to \infty} \frac{1}{T'} \ln\langle\theta|W|\theta\rangle \tag{63}$$

We can write this expectation value as a path integral:

$$\langle\theta|W|\theta\rangle = \frac{\int [dA][d\psi][d\psi^*] W e^{-S} e^{i\nu\theta}}{\int [dA][d\psi][d\psi^*] e^{-S} e^{i\nu\theta}} \tag{64}$$

The denominator is just what we analyzed in Eq.(48) to write the energy density:

$$\langle\theta|e^{-HT}|\theta\rangle = \exp\left(2KLTe^{-S_0}\cos\theta\right), \tag{65}$$

The numerator can be split into a part inside the loop and one outside the loop. The outside part is just the denominator integral except that the available spacetime area is $(LT - L'T')$. In the numerator we can write using Eq.(59),

$$W = \exp\left(2\pi i q \nu^{\text{inside}}/e\right) \tag{66}$$

where ν^{inside} is the contribution to the winding number from finite field configurations within the loop of area $L'T'$. Hence the contribution from this part is similar to the previous case but with '$\theta + 2\pi q/e$' instead of 'θ' and with spacetime area $L'T'$.

Thus we can write for the expectation value:

$$\ln\langle\theta|W|\theta\rangle = 2Ke^{-S_0}\left[(LT - L'T')\cos\theta + L'T'\cos(\theta + 2\pi q/e) - LT\cos\theta\right] \tag{67}$$

where the three terms come from the inside loop numerator, outside loop numerator and the denominator respectively. Hence using Eq.(63),

$$\Delta = 2L'Ke^{-S_0}\left[\cos\theta - \cos(\theta + 2\pi q/e)\right] \tag{68}$$

We see that the energy shift due to the presence of external test charges is directly proportional to the distance between them. Thus the force between them, which is the gradient of the energy shift, is constant, as we mentioned in the beginning. Hence we conclude that $\mu^2 < 0$ does not really affect the theory as instantons rescue us and keep the mesons confined. But Δ has an $e^{-1/\hbar}$ (we had set $\hbar = 1$ but S_0 is always divided by $\hbar$.) dependence which does not arise in the positive μ^2 case.

APPENDIX A: GAUGE THEORY CONVENTIONS

We will adopt the convention of taking the Lie algebra generators to be anti-Hermitian,

$$[T^a, T^b] = f^{abc}T^c \tag{A1}$$

and the Cartan inner product is

$$(T^a, T^b) = \delta^{ab}. \tag{A2}$$

The gauge fields A_μ are taken as matrix-valued vector fields in the adjoint representation of the Lie algebra,

$$A_\mu = gA^a_\mu T^a, \tag{A3}$$

where g is the gauge coupling. The field strength tensor in this notation is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{A4}$$

The action in Euclidean space is given by

$$S = \frac{1}{4g^2} \int d^4x (F_{\mu\nu}, F_{\mu\nu}) \equiv \frac{1}{4g^2} \int F^2. \tag{A5}$$

The gauge transformations are represented as

$$g(x) = e^{\lambda^a(x) T^a}. \tag{A6}$$

Under such a gauge transformation,

$$A_\mu \to g A_\mu g^{-1} + g \partial_\mu g^{-1} \tag{A7}$$

and

$$F_{\mu\nu} \to g F_{\mu\nu} g^{-1} \tag{A8}$$

The classical equation of motion for this theory is

$$D_\mu F_{\mu\nu} = 0, \tag{A9}$$

where D_μ is the covariant derivative whose action on the field strength is

$$D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]. \tag{A10}$$