Batalin-Vilkovisky Formalism

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1 Introduction

The fundamental interactions of nature are governed by gauge theories. Gauge symmetries necessarily imply redundant description from the point of view of dynamics. Although redundant degrees of freedom can be eliminated in a number of ways there are reasons of not doing so. Among them there are necessity for the manifestation of covariance, locality of interactions etc. Quantization generally requires introduction of ghost fields. In linear gauges in electrodynamics ghosts decouple and can be ignored. In non-abelian theories we require interacting ghosts. A major progress in understanding was the Fadeev-Popov procedure. Batalin-Vilkovisky formalism is a generalization of Fadeev-Popov that allows us to deal with a class of more general theories: theories with reducible gauge algebras.

2 Preliminaries

Let's consider a system whose dynamics is governed by a classical action $S_0[\phi]$. By ϕ we will be denoting a set of fields $\phi^i(x), i = 1, \ldots, n$. The index i can denote index of arbitrary nature: spacetime indices of tensor fields, spinor indices of fermion fields etc. Let $\epsilon(\phi^i) = \epsilon_i$ denote the statistical parity of ϕ^i. Each field ϕ^i can be either commuting ($\epsilon_i = 0$) or anticommuting ($\epsilon_i = 1$) with naturally defined generalized commutator $\phi^i(x)\phi^j(y) = (-1)^{\epsilon_i\epsilon_j}\phi^j(y)\phi^i(x)$.

2.1 Right and left derivatives

Left and right derivatives are defined correspondingly:

$$\frac{\partial_l X}{\partial\phi} \equiv \frac{\vec{\partial}}{\partial\phi} X \quad \frac{\partial_l X}{\partial\phi} \equiv X \frac{\overleftarrow{\partial}}{\partial\phi} \tag{1}$$

Right derivatives act from right to left. The differential $dX(\phi) = d\phi \frac{\partial_l X}{\partial\phi} = \frac{\partial_r X}{\partial\phi} d\phi$. Now let's establish the connection between left and right derivatives. We may assume that $X = \phi Y + Z$. Where without loss of generality Y and Z have no ϕ dependence. The left and the right derivatives of X are then

$$\partial_l X/\partial\phi = Y \quad \partial_r X/\partial\phi = (-1)^{\epsilon_Y} Y = (-1)^{\epsilon(\phi)(\epsilon_X+1)} Y \tag{2}$$

So for all cases

$$\partial_r X/\partial\phi = (-1)^{\epsilon_Y} Y = (-1)^{\epsilon(\phi)(\epsilon_X+1)} Y \tag{3}$$

2.2 Reducible and irreducible gauge theories

The simplest gauge theories, for which all gauge transformations are independent, are called irreducible. When some gauge transformations are dependent such gauge theories are called reducible. In reducible gauge theories there is a kind of "gauge invariance" for gauge transformations, they are referred to as "level-one" gauge invariances. Also "level-two" gauge invariances can occur, i.e. gauge invariances for level-one invariances etc. The generalization of this idea is the concept of so called *Lth stage reducible theory.*

Let's assume that the action is invariant under a set of m_0 ($m_0 \leq n$) non-trivial gauge transformations, which read in infinitesimal form

$$\delta\phi^i(x) = [R^i_\alpha(\phi)\varepsilon^\alpha](x) \tag{4}$$

Each gauge parameter is either commuting $\epsilon(\varepsilon^\alpha) = \epsilon_\alpha = 0$ or anticommuting $\epsilon_\alpha = 1$. The statistical parity of R^i_α is $\epsilon(R^i_\alpha) = \epsilon_i + \epsilon_\alpha$ (mod 2). Let $S_{0,i}(\phi, x)$ denote the variation of the action with respect to $\phi^i(x)$:

$$S_{0,i}(\phi, x) = \frac{\partial_r S_0[\phi]}{\partial\phi^i(x)} \tag{5}$$

The statement that the action is invariant under the gauge transformations (for arbitrary values of gauge parameters within some range) means that the Noether identities hold:

$$S_{0,i}R^i_\alpha = 0 \tag{6}$$

To commence a perturbation theory, one searches for the solutions to the classical equations of motion, $S_{0,i}(\phi, x) = 0$, and then expands about these solutions. The theory is based on the assumption that there exists at least one stationary point $\phi_0 = \phi_0^j$ such that

$$S_{0,i}|_{\phi_0} = 0. \tag{7}$$

This equation defines a surface Σ in function space, which is infinite dimensional when gauge symmetries are present.

As a consequence of the Noether identities, the equations of motion are not independent, propagators do not exist.

The most general solution to the Noether identities is a gauge transformation, up to terms proportional to the the equations of motion:

$$S_{0,i}\lambda^i = 0 \Leftrightarrow \lambda^i = R^i_{0\alpha_0}\lambda'^{\alpha_0} + S_{0,j}T^{ji} \tag{8}$$

The subscript 0 on the gauge generator indicates the level of gauge transformation. The second term $S_{0,j}T^{ij}$ is known as a trivial gauge transformation.

If the functionals $R^i_{0\alpha_0}$ are independent on-shell then the theory is *irreducible*. In this case

$$rank\ R^i_{0\alpha_0}|_\Sigma = m_0 \tag{9}$$

However there might be dependencies among the gauge generators left and, as a consequence the rank of the generator might be less than their number. Precisely if $rank R^i_{0\alpha_0}|_\Sigma < m_0$ the theory is reducible. If $m_0 - m_1$ of the generators are independent on-shell, then there are m_1 relations among them and there exists m_1 functionals $R^{\alpha_0}_{1\alpha_1}$ such that

$$R^i_{0\alpha_0}R^{\alpha_0}_{1\alpha_1} = S_{0,j}V^{ji}_{1\alpha_1}, \quad \alpha_1 = 1, \ldots, m_1; \quad \epsilon(R^{\alpha_0}_{1\alpha_1}) = \epsilon_{\alpha_0} + \epsilon_{\alpha_1}\ (mod\, 2) \tag{10}$$

The $R^{\alpha_0}_{1\alpha_1}$ are the on-shell null vectors for $R^i_{0\alpha_0}$ since $R^i_{0\alpha_0}R^{\alpha_0}_{1\alpha_1}|_\Sigma = 0$ on-shell. If the functionals $R^{\alpha_0}_{1\alpha_1}$ are independent on-shell

$$rank\ R^{\alpha_0}_{1\alpha_1}|_\Sigma = m_1, \tag{11}$$

then the theory is called *first-stage reducible*. The generalization of the this leads to Lth stage reducible theory.

2.3 Abelian gauge theories

Let's have a look at the example of an Lth stage reducible theory. Let A be a p-form and define F to be its field strength: $F = dA$ where d is the exterior derivative. For $p+1$ dimension d of spacetime, an action for this theory is

$$S_0 = -\frac{1}{2}\int F \wedge *F, \tag{12}$$

where $*$ is the Hodge star taking a q-form to a $d-q$-form and $\wedge$ is the wedge product. Using the nilpotency of exterior derivative ($d^2 = 0$) one sees that the action is invariant under the gauge transformation

$$\delta A = d\lambda_{p-1}, \tag{13}$$

where λ_{p-1} is a $p-1$-form. The gauge transformation has its own gauge invariance which is, not surprisingly, is a derivative of a $p-2$-form etc:

$$\delta\lambda_{p-1} = d\lambda_{p-2}, \ldots, \delta\lambda_1 = d\lambda_0, \tag{14}$$

where λ is a q-form. The number of degrees of freedom is

$$n_{dof} = C_d^p - C_d^{p-1} + C_d^{p-2} - \cdots + (-1)^p C_d^0 = C_{d-1}^p \tag{15}$$

C_d^q is the number of combination and naturally the dimension of the space of q-forms in d dimensional spacetime.

The gauge generators at the sth stage, $R_{s\alpha_s}^{\alpha_s - 1}$, correspond to the exterior derivative d acting on the space of $(p-1-s)$ forms, $d^{(p-1-s)}$:

$$R_0 \leftrightarrow d^{(p-1)}, R_1 \leftrightarrow d^{(p-2)}, \ldots, R_{p-1} \leftrightarrow d^{(0)} \tag{16}$$

3 The classical formalism

Here we present the formalism at the classical level. It involves five steps:

The original configuration space is enlarged to include additional fields such as ghost fields, ghosts for ghosts etc. Antifields for all those fields are also introduced.

On the space of fields and antifields an odd symplectic structure (,) is defined which is referred to as "antibracket".

The classical action S_0 is extended to inlcude terms involving ghosts and antifields and is denoted by S.

The *classical master equation* is defined to be (S,S) = 0.

Solutions to the classical master equation subject to certain boundary conditions can be found.

Suppose we have an irreducible theory with m_0 gauge invariances. Then m_0 ghost fields are needed. In this case the field set is $\Phi^A = \{\phi^i, C_0^{\alpha_0}\}$, where $\alpha_0 = 1, \ldots, m_0$. If the theory is Lth stage reducible the set of fields is $\Phi^A = \{\phi^i, C_s^{\alpha_s}; s = 0, \ldots, L; \alpha_s = 1, \ldots, m_s\}$. To each of these fields an additive conserved charge, called ghost number, is assigned. The classical fields ϕ_i are assigned ghost number zero, ordinary ghosts have ghost number 1, level-one ghosts have ghost number two, etc. Similarly, ghosts have opposite statistics of the corresponding gauge parameter, ghosts for ghosts meanwhile have the same statistics as the gauge parameter and so on. We write this as following:

$$gh[C_s^{\alpha_s}] = s+1, \quad \epsilon(C_s^{\alpha_s}) = \epsilon_{\alpha_s} + s + 1 \, mod \, 2 \tag{17}$$

Next step is introduction of antifields Φ_A^*, $A = 1, \ldots, N$ for each field Φ_A. The ghost number and statistics of Φ_A^* are

$$gh[\Phi_A^*] = -gh[\Phi_A] - 1, \quad \epsilon(\Phi_A^*) = \epsilon_{\Phi_A} + 1 \, mod \, 2 \tag{18}$$

So the statistics of Φ_A^* is opposite to that of Φ_A.

3.1 Antibracket

In the space of fields and antifields the antibracket is defined by

$$(X, Y) \equiv \frac{\partial_r X}{\partial \Phi^A} \frac{\partial_l Y}{\partial \Phi_A^*} - \frac{\partial_r X}{\partial \Phi_A^*} \frac{\partial_l Y}{\partial \Phi^A}. \tag{19}$$

Many properties of (X, Y) are similar to a graded Poison bracket, with the grading of X and Y being $\epsilon_X + 1$ and $\epsilon_Y + 1$ instead of ϵ_X and ϵ_Y. The antibracket satisfies

$$
\begin{gathered}
(Y,X) = -(-1)^{(\epsilon_X+1)(\epsilon_Y+1)}(X,Y),\\
((X,Y),Z) + (-1)^{(\epsilon_X+1)(\epsilon_Y+\epsilon_Z)}((Y,Z),X) + (-1)^{(\epsilon_Z+1)(\epsilon_X+\epsilon_Y)}((Z,X),Y) = 0,\\
gh[(X,Y)] = gh[X] + gh[Y] + 1,\\
\epsilon[(X,Y)] = \epsilon_X + \epsilon_Y + 1\ (mod\, 2).
\end{gathered}
\tag{20}
$$

For example let's prove the first property :

$$
\begin{aligned}
&(Y,X) = \tfrac{\partial_r Y}{\partial \Phi^A}\tfrac{\partial_l X}{\partial \Phi^*_A} - \tfrac{\partial_r Y}{\partial \Phi^*_A}\tfrac{\partial_l X}{\partial \Phi^A}\\
&= (-1)^{(\epsilon_X+\epsilon_A)(\epsilon_X+\epsilon_A+1)}\tfrac{\partial_l X}{\partial \Phi^*_A}\tfrac{\partial_r Y}{\partial \Phi^A} - (-1)^{(\epsilon_X+\epsilon_A)(\epsilon_Y+\epsilon_A+1)}\tfrac{\partial_l X}{\partial \Phi^A}\tfrac{\partial_r Y}{\partial \Phi^*_A}
\end{aligned}
\tag{21}
$$

Now using equation (2) we obtain

$$
(-1)^{\epsilon_X\epsilon_Y+\epsilon_X+\epsilon_Y+1}\frac{\partial_r X}{\partial \Phi^*_A}\frac{\partial_l Y}{\partial \Phi^A} - (-1)^{\epsilon_X\epsilon_Y+\epsilon_X+\epsilon_Y+1}\frac{\partial_r X}{\partial \Phi^A}\frac{\partial_l Y}{\partial \Phi^*_A} = -(-1)^{(\epsilon_X+1)(\epsilon_Y+1)}(X,Y).
\tag{22}
$$

The antibracket carries ghost number one and has odd statistics.

It also has the following properties

$$
\begin{aligned}
&(B,B) = 2\tfrac{\partial_r B}{\partial \Phi^A}\tfrac{\partial_l B}{\partial \Phi^*_A}, \quad (F,F) = 0,\\
&((X,X),X) = 0
\end{aligned}
\tag{23}
$$

3.2 Classical master equation

Let $S[\Phi,\Phi^*]$ be an arbitrary functional of fields and antifields with dimensions of action and with ghost number zero and even statistics: $\epsilon(S) = 0$ and $gh[S] = 0$. The equation

$$
(S,S) = 2\frac{\partial_r S}{\partial \Phi^A}\frac{\partial_l S}{\partial \Phi^*_A}
\tag{24}
$$

is called the *classical master equation*.

The solution of master equation should satisfy certain boundary conditions. A relevant solution plays a double role. On one hand, a solution S is the generating functional for the structure functions of the gauge algebra. On the other hand, S is the stating action to quantize covariantly the theory.

The variations of S with respect to Φ^A and Φ^*_A are the equations of motion:

$$\frac{\partial_r S}{\partial z^a} = 0 \tag{25}$$

where we use collective variables z^a. Equations of motion define a surface Σ in the full space of fields and antifields. Restriction to this surface is what is meant by being "on-shell".

An action S, satisfying the master equation, possesses its own set of gauge invariances. Indeed, by differentiating it with respect to z^b, one finds that

$$\tfrac{\partial_r S}{\partial z^a} R^a_b = 0, \quad a = 1, ..., 2N R^a_b \equiv \zeta^{ac} \tfrac{\partial_l \partial_r S}{\partial z^c \partial z^b}. \tag{26}$$

Although there appears to be $2N$ gauge invariances, not all of them are independent on-shell. Differentiating with respect to z^d, multiplying by ζ^{cd}, using the definition of R^a_b and imposing stationary condition we find

$$R^c_a R^a_b|_\Sigma = 0 \tag{27}$$

The matrix R^a_b turns out to be nilpotent on-shell. A nilpotent $2N \times 2N$ matrix has rank less or equal to N. Hence at a stationary point there exist at least N relations among the gauge generators R^a_b and therefore the number $2N - r$ of independent gauge transformations on-shell is greater or equal to N, where r is the rank of the hessian of S at the stationary point:

$$r \equiv rank \frac{\partial_r \partial_l S}{\partial z^a \partial z^b}|_\Sigma \tag{28}$$

A solution to the master equation is called proper if $r = N$.

Now let's specify the relation between S_0 and S. To make contact with the original theory, one requires the proper solution to contain the original action $S_0[\phi]$. This requirement ensures the correct classical limit. It corresponds to the following boundary condition on S

$$S[\Phi, \Phi^*]|_{\Phi^*} = S_0[\phi] \tag{29}$$

An additional boundary requirement is

$$\frac{\partial_r \partial_l S}{\partial C^*_{s-1,\alpha_{s-1}} \partial C^{\alpha_s}_s}|_{\Phi^*=0} = R^{\alpha_{s-1}}_{s\alpha_s}(\phi), \quad s = 0, \ldots, L. \tag{30}$$

For notational convenience we define

$$C^{\alpha_{-1}}_{-1} \equiv \phi^i,\ C^*_{-1,\alpha_{-1}} \equiv \phi^*_i,\ \alpha_{-1} = i. \tag{31}$$

Given the ghost number restriction and the boundary conditions the expansion necessarily has the following form:

$$S[\Phi, \Phi^*] = S_0[\phi] + \Sigma^L_{s=0} C^*_{s-1,\alpha_{s-1}} R^{\alpha_{s-1}}_{s\alpha_s} C^{\alpha_s}_s + O(C^{*2}). \tag{32}$$

3.3 The classical BRST symmetry

Let's define the classical BRST symmetry transformation of a functional X of fields and antifields by

$$\delta_B X \equiv (X, S). \tag{33}$$

The transformation of fields is therefore

$$\delta_B \Phi^A = \frac{\partial_l S}{\partial \Phi^*_A} \tag{34}$$

$$\delta_B \Phi^*_A = -\frac{\partial_l S}{\partial \Phi_A} = (-1)^{\epsilon_A + 1} \frac{\partial_r S}{\partial \Phi^A} \tag{35}$$

The field anti-antifield action is classically BRST symmetric

$$\delta_B S = 0 \tag{36}$$

what follows from $(S, S) = 0$.

The BRST operator δ_B is a nilpotent graded derivation:

$$\delta_B(XY) = X\delta_B Y + (-1)^{\epsilon_Y}(\delta_B)Y \tag{37}$$

$$\delta^2_B X = 0. \tag{38}$$

The nilpotency follows from two properties of the antibracket : the graded Jacobi identity and graded antisymmetry.

$$((X, S)S) = -((S, S), X) + (-1)^{\epsilon_X + 1}((S, X), S) = -((S, S), X) - ((X, S), S), \tag{39}$$

which leads to to $((X, S), S) = -\frac{1}{2}((S, S), X) = 0$.

A functional O is a classical observable if $\delta_B O = 0$ and $O \neq \delta_B Y$ for some Y. Two observables are equivalent if they differ by a BRST transformation.

3.4 Abelian p-form theories

This is an example of $(p-1)$-stage off-shell reducible theories. Consequently there are p different types of ghosts: $C_0, C_1, \ldots, C_{p-1}$, where C_s is a $(p-1-s)$-form. So $\Phi^A = A, C_0, C_1, ..., C_{p-1}$, $\Phi^*_A = A^*, C^*_0, C^*_1, ..., C^*_{p-1}$.

The proper solution of classical master equation is

$$S = \int -\frac{1}{2} F \wedge * + *(A*) \wedge dC_0 + \sum_{i=1}^{p-1} *(C^*_{i-1}) \wedge dC_i \tag{40}$$

As a consequence of integration by parts and $d^2 = 0$ $(S, S) = 0$:

$$\frac{\partial_r S}{\partial A_{\mu_1 \mu_2 \ldots \mu_p}} \frac{\partial_r S}{\partial A^{*\mu_1 \mu_2 \ldots \mu_p}} \propto \int (*d * F) \wedge *dC_0 \propto \int d(*F \wedge dC_0) = 0, \tag{41}$$

$$\frac{\partial_r S}{\partial (C_i)_{\mu_1 \mu_2 \ldots \mu_{p-i-1}}} \frac{\partial_r S}{\partial (C_i)^{*\mu_1 \mu_2 \ldots \mu_{p-i-1}}} \propto \int (*d * (C^*_i) \wedge *dC_{i+1} \propto \int d(*F \wedge dC_{i+1}) = 0, \tag{42}$$

4 Quantum master equation

First of all let's introduce antifields Φ^*_A and look for such an action $W(\Phi, \Phi^*)$ that

$$\Phi^*_A | \Sigma = \frac{\partial_r \Psi(\Phi)}{\partial \Phi^A} \tag{43}$$

Let's have a look at vacuum-vacuum amplitude:

$$Z_\Psi = \int \prod d\Phi^A exp\{\frac{i}{\hbar} W(\Phi, \frac{\partial_r \Psi}{\partial \Phi})\} \tag{44}$$

When we shift Ψ by a $\delta\Psi$ the amplitude changes by

$$\delta Z = i \int \prod d\Phi^A exp\{\frac{i}{\hbar} W(\Phi, \frac{\partial_r \Psi}{\partial \Phi})\} \frac{\delta_r W(\Phi, \Phi^*)}{\delta \Phi^*_A} |_{\Phi^*_A = \delta\Psi/\delta\Phi} \frac{\delta(\delta\Psi)}{\delta \Phi^A} \tag{45}$$

By means integrating by parts we obtain the following equivalent of classical master equation:

$$(W, W) = i2\hbar \Delta W, \quad \Delta \equiv \frac{\partial_r}{\partial \Phi^A} \frac{\partial_l}{\partial \Phi^*_A} \tag{46}$$

This equation is equivalent to

$$\Delta exp\left[\frac{i}{\hbar}W\right]=0 \tag{47}$$

If we present W as

$$W=S+\sum_{p=1}^{\infty}\hbar^p M_p, \tag{48}$$

where S is the classical part of the action, and the reminder is the contribution of the quantum integration measure, which secures the invariance of the functional integral. After substitution of this expansion to equation (46) we find:

$$\begin{array}{ll} (S,S)=0, & (M_1,S)=i\Delta S \\ (M_p,S)=i\Delta M_{p-1}-\frac{1}{2}\sum_{q=1}^{p-1}(M_q,M_{p-q}), & p\geq 2. \end{array} \tag{49}$$

Further consideration leads to construction of the solution of the master equation in the explicit form according to certain boundary conditions.

5 Conclusion

Batalin-Vilkovisky formalism is a convenient instrument used for analysis of possible symmetry-breaking by quantum effects. It allows us to work with unclosed or irreducible gauge algebras where other methods fail. The presented formalism introduces ghosts from the outset and automatiaclly incorporates the BRST symmetry. In a very direct sense it is a generalization of Fadeev-Popov quantization procedure.

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